

V. I. Ivanenko

**Decision Systems  
and Nonstochastic  
Randomness**

 Springer

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*In memory of A. A. Feldbaum*



In contemplation, if a man begins with certainties  
he shall end in doubts; but if he be content to  
begin with doubts, he shall end in certainties.

—*Francis Bacon*



# Preface

This book unifies the results of joint studies of the author and his younger colleague, Valery A. Labkovsky, now deceased. Whenever reference is made to the author of this book, the reader should understand this as *authors*.

The notion of “decision system,” analogous to the notion of “control system” used in control theory, has served as the basis for this unification. This analogy was noticed by A. A. Feldbaum in the 1960s, and his views influenced both of us greatly at the time.

Nevertheless, this analogy should not serve as an impediment to a realization of the main difference between decision theory and control theory: in decision theory, the nature of the criterion of the best decision is linked to the psychology of the decision-maker, and that is its central problem.

Perhaps this is why decision theory pays so much attention to the problem of uncertainty. It turns out that the notion of a random-in-a-broad-sense phenomenon proposed by A. N. Kolmogorov, which divides the whole spectrum of random phenomena into two classes, stochastic and nonstochastic, becomes most appropriate in this context. However, it is natural to ask whether nonstochastic phenomena exhibit any regularity? A positive answer to this question—the theorem of existence of so-called statistical regularities—was the second reason for the union in this book of various results of our joint studies.

The author thanks Anatoly Skorokhod for many years of courtesy and support of our research in the insufficiently studied field of nonstochastic random phenomena; Bertrand Munier for the suggestion that this book be published in English; Stephan H. Crandall, who was one of the imaginary readers of this book, for help in visiting the United States for research in university libraries; Yaroslav Ivanenko and Vadym Mikhalevich for useful discussions, remarks, and suggestions; Ludmila and Yaroslav Ivanenko for their patience and assistance in preparation of this book in electronic format.

And finally, it is I alone, Victor Ivanenko, who is responsible for any flaws or other shortcomings in this book. I will be grateful to the reader for any suggestions or corrections.



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# Chapter 1

## Introduction

There are some questions that never age. It is answers that grow old. We begin our book with such a question. “Four kings—Hanoverians—George I, George II, George III, and George IV all died on the same day of the week, Saturday. The probability that we are dealing here with a random event is extremely low. Can we make a conclusion that Saturday is a day of ill omen for this dynasty? And if so, then what is the value of statistical methods of hypothesis verification?”

Discussion around this question took place on the front pages of the journal *Nature* in 1968, and the above quotation was taken from [66].

In that book there is another approach to statistical verification of the hypothesis: if we take into account that there were 53 kings in the Hanoverian dynasty, then the probability of four kings with the same name dying on the same day of the week is equal to only  $1/7$ .

Leaving aside the question how the author obtained this number (he probably used some additional information not mentioned in the text), we shall focus our attention on the following fundamental fact: the quantitative result that we obtain in a statistical problem depends dramatically on our viewpoint.

To eliminate any doubt about the objective character of this observation, let us take another example, this time from a textbook on probability theory by the extraordinary mathematician and teacher W. Feller [19]. Here is problem number 23 from Chapter 11 of Feller’s book: *The Family Problem*: In a certain family four sisters do the dishes. Four plates are broken, three of them by the youngest sister. Therefore she is considered awkward. Can we justify her action by attributing the broken dishes to chance?

There is a solution at the end the book: “Assuming that chance is responsible, the probabilities that three of four plates are broken by (a) one girl and (b) the youngest girl are equal to  $13/64 \approx 0.2$  and  $13/256 \approx 0.05$  respectively.” Again, we see that the more precise the description of what happened is, the smaller the probability this event has, and, thus, the less ground we have to consider this event as random. Again, the conclusions depend on the hypothesis taken (so, can we justify the youngest sister or no?).

One may agree with the author of [66] that “when making a hypothesis liable to statistical verification it is necessary to take into account common sense.” But later on, he states that the hypothesis of the ominous interpretation of Saturday for the four Hanoverian Georges is “ridiculous,” but this conclusion does not seem so obvious.

Indeed, imagine that, say, George IV is very sick, and he knows that George I, George II, and George III all died on a Saturday. On which day of the week is he more likely to expect to die? After all, a person’s health and wellbeing may depend considerably on his or her emotional state.

In the *Family Problem*, the validity of both hypotheses is even more obvious. Thus, common sense is really necessary in making statistical hypotheses, but it is not sufficient. Let us illustrate these considerations by another example, from another field.

What is the essence of the life experience of a human being (or other animal)? It is likely that interaction with the environment leaves in one’s memory certain links between events. For example, in Pavlov’s classical experiments on the formation of a conditional reflex in a dog, a researcher repeatedly offers food to the accompaniment of a ringing bell. Finally, the dog begins to salivate on hearing only the bell. Describing the experiment in this way, we consider the following events: (a) the appearance of food; (b) the ringing of the bell; (c) secretion of saliva. Event (c) appears in response to event (b), and this confirms the fact that the link between (a) and (b) has indeed been established. But why, along with (a) and (b), do we not take into account the time of day, for example, or the temperature of the air, the physiological condition of the dog, the researcher’s body odor?

In probability theory there is a notion of an elementary, i.e., “indivisible,” event, in contrast to an event that consists of a set of several elementary events. In real life, we are never able to state with absolute assurance that an event is elementary. So, if the introduction of food is an event for the researcher, for the dog it may be the introduction of food of a certain quality; if for one statistician the event is that all plates are broken by one sister, for another it may be that they are broken by the youngest one; and so on.

At first glance, the situation seems to be similar to scenarios from other fields of human endeavor: we never know anything for certain, and so we use approximate knowledge. Nevertheless, here there is a principal difference.

Imagine that we have finally found the true first principles and thus have succeeded in describing, following Laplace, all the coordinates and velocities of all the particles of the universe. We may try to consider these points of the phase space as elementary events and thus describe all our experience in these terms. Immediately, we see that the value of our experience will be zero, since none of these elementary events will be ever repeated. This means that even if we were able to describe and memorize our experience exactly, we would be able to use it only after a certain coarse-graining to some other events (or notions).

So what can we conclude?

1. In any statistical research, or even more generally in any conscious activity, in any reasoning, we deal with events none of which is elementary.
2. Events that we consider predetermine the conclusions that we obtain.
3. The reasons leading to the consideration of this or that event are linked, in each particular case, with what is called “common sense.” However, “common sense” differs from person to person. Moreover, even when we consider a professional activity of an outstanding specialist in his field (William Feller), even in this case common sense does not guarantee the uniqueness of conclusions.

This range of questions stimulated our research, the results of which are presented in this book. In particular, these results include the following (Theorem 6.5). Any system of events (or notions) is optimal or not optimal only with respect to a certain “decision system,” that is, only when, generally speaking, we know the goal that we would like to reach, as well as the tools that we can use. Moreover, for any decision system we can indicate an optimal system of notions (events), and conversely, any system of notions (events) is optimal for a certain decision system. The notion of a decision system is used here analogously to the notion of a control system in control theory. Like a control system, a *decision system* is a pair formed by a *decision-maker* and a *decision situation* in which the decision-maker finds herself in a *situation* demanding that she choose a single decision (action) from some set of possible decisions. Only within a decision system does the *decision problem*, consisting of a mathematical model of the decision situation and the decision criteria, make sense. Such a formulation of decision-making does not differ from that used in economics, which is where decision theory began its development [69, 73, 20, 58, 8]. At the same time, this formulation and terminology go beyond the limitations of decision theory applications in economics.

Let  $\Theta$  be a set of possible values of some parameter  $\theta$ ,  $U$  a set of possible decisions  $u$ , and  $L : \Theta \times U \rightarrow R$  a bounded real “loss” function. Our aim consists in the choice of  $u \in U$  that minimizes the loss function  $L$ .

It is clear that the solution to this problem depends on what we know about the parameter  $\theta \in \Theta$ . For instance, if parameter  $\theta$  assumes the specific value  $\theta^*$ , we are dealing with the classical extremal problem

$$L(\theta^*, u) \rightarrow \inf \quad (u \in U).$$

We say that in such a problem there is no uncertainty, and the problem is called a *deterministic optimization problem*.

If nothing is known about  $\theta$ , then maximal losses

$$\sup_{\theta \in \Theta} L(\theta, u) \rightarrow \inf \quad (u \in U)$$

are sometimes minimized. This case is usually called *complete* or *strict uncertainty* [21].

It is clear that both of these problems in some sense are polar opposites. In practical situations, we rarely know precisely the value of the parameter, but even more rarely do we know nothing about it.

The real world is somewhere between these two poles. But how many models of this “world of uncertainty” do we have today? Obviously not many. There are only a few points, a few models in between these poles.

One such well-developed model encompasses the case in which the unknown parameter  $\theta$  is stochastically random with a given distribution  $\mu$ . In this case,  $u$  is chosen to minimize mean losses

$$\int L(\theta, u)\mu(d\theta) \rightarrow \inf \quad (u \in U).$$

This problem is known as a *stochastic* or *Bayesian decision problem* [17]. The stochastic decision problem is often strongly criticized because in practice, one rarely knows the distribution.

The so-called hybrid presentation of uncertainty proves to be somewhat more flexible. In this case, one supposes that the distribution of the parameter  $\theta$  is known to within a distribution from a given class  $\mathcal{M}$  of distributions. This kind of uncertainty is sometimes called *stochastic uncertainty*. In this case, the decision problem usually has the form

$$\sup_{\mu \in \mathcal{M}} \int L(\theta, u)\mu(d\theta) \rightarrow \inf \quad (u \in U).$$

It is clear that if the set  $\mathcal{M}$  consists of a single element  $\mu$ , the decision problem with stochastic uncertainty becomes a Bayesian decision problem. If the set  $\mathcal{M}$  consists of all possible distributions on  $\Theta$ , then there is a temptation to call such uncertainty complete or total. But this is not entirely correct. In fact, the assumption that the probability distribution belongs to a certain class  $\mathcal{M}$  is not as harmless as it seems. What does it mean that a certain variable has a probability distribution? At the least, it means that the frequencies to get into certain subsets tend to certain limits (probability theory demands more, but here we shall concentrate only on this fact). However, can we conceive of an experiment or of a line of reasoning that would show that frequencies tend to their limits yet at the same time not show to which limits exactly?

It seems that such is possible in only one situation: when we know a fixed set of “random mechanisms” of which one and only one will be used to generate the random variable. For example, we know that a coin or a die, likely asymmetric, will be tossed. In this case, it is clear that a simple change of notation leads us to complete uncertainty, but this time with respect to the “random mechanism.”

There is a more realistic situation in which not only the distribution is unknown, but also whether one can describe the behavior of the parameter  $\theta$  by means of a probability distribution. However, even in this case there are no good reasons to say that we are dealing with complete uncertainty.

When we call uncertainty complete, we do not exclude every conceivable behavior of the unknown parameter. In particular, we do not exclude that its value is chosen by a conscious opponent, an antagonist, fully aware of our actions. That is why we wish to minimize maximal losses. But in real life, in the majority of practical decision-making situations, there is no conscious opponent, and we are fully aware of that. In [80] we find the following comment: “It seems that the absence of a conscious antagonist facilitates the decision-making problem. It turns out that no, not only does it not facilitate, but on the contrary, it complicates the problem. Indeed, it is ‘easier’ to succeed in the ‘game against Nature’ (there is no antagonist), but it is more difficult *to justify one’s choice.*”

This difficulty might be overcome if we knew how to describe the behavior of the unknown parameter in the absence of conscious opposition, which we call *indifferent uncertainty*. Such a description is proposed in Chapter 3. It turns out that indifferent uncertainty, in a variety of situations, may be viewed as stochastic uncertainty. But not always. For this reason, as well as for some other reasons, it is necessary to construct a more general mathematical formalism in order to describe the behavior of the unknown parameter. This necessity was indicated already by E. Borel in [4]. A. N. Kolmogorov, in [47], proposed to call such behavior “random in a broad sense.” This is considered in Chapter 4, where we introduce the notion of *statistical regularity*, generalizing the notion of probability distribution. It is proved (Theorem 4.2) that any *sampling directedness* has a statistical regularity and any statistical regularity corresponds to certain sampling directedness. It turns out that the concept of sampling directedness relates to the concept of statistical regularity, just as in probability theory, the concept of random value relates to the concept of probability distribution. In Chapter 4 there are also some examples of statistical regularity for random (in a broad sense) phenomena, in particular, the regularity of a statistically unstable sequence (Theorems 4.3 and 4.4).

The same notion of statistical regularity appears from entirely different considerations in Chapter 5, dealing with the study of a link between the information about the behavior of the unknown parameter  $\theta$  and the criterion that determines the choice of a decision  $u$  in a decision system. It turns out that if one chooses the guaranteed result principle<sup>1</sup> generalized for mass phenomena as the optimality principle, then, under quite broad assumptions, the decision problem is given by the quadruple  $S = (\Theta, U, L, \mathbf{P})$ , which is the mathematical model of a decision situation, and by the function

---

<sup>1</sup> This principle seems to be the most natural form of uncertainty aversion [37, 38].

$$L^*(u) = \sup_{p \in \mathbf{P}} \int L(\theta, u) p(d\theta) \rightarrow \inf \quad (u \in U),$$

which is the decision criterion, where  $\mathbf{P}$  is the statistical regularity on  $\Theta$  in the form of a closed, in the corresponding topology, nonempty family of finitely additive distributions.<sup>2</sup> We call it the *general decision problem* (Theorem 5.2). The decision problem with indifferent uncertainty, the decision problem with complete uncertainty, the stochastic decision problem, and the decision problem with stochastic uncertainty turn out to be special cases of the general decision problem.

Chapters 6, 7, and 8 constitute the second part of the book. This part deals with the study of experiment in decision systems and the measurement of information supplied by experimentation. In Chapter 6 we show that the notion of decision system can be used to measure the informativity (information quantity, usefulness, value) of an arbitrary observation on  $\theta$ . More precisely, we define the subdivision of a set on equivalent classes as a classification. We say that some classification is optimal for a fixed decision problem when

1. in order to make the optimal decision in a given decision problem it is sufficient to know  $\theta$  only to within its equivalence class;
2. no classification with a smaller quantity of classes possesses this property.

We prove that an optimal classification exists for any decision problem and each classification is optimal for some decision problem.

In Chapter 7 it is proved that informativity of experiment (or observation), a notion introduced in Chapter 6, is, in some sense, the only possible one. In other words, there is no method to measure the quantity of information other than its value [62] or usefulness (Theorem 7.1). In particular, Shannon's "quantity of information" turns out to be the informativity or usefulness of experiment with respect to a special decision problem.

In Chapter 8 we consider an application of some of our results to the multistep decision problem [14, 18] that triggered all the research undertaken in this book.

Chapter 2 occupies a special place. It addresses mainly newcomers to the field, but it may be read by specialists as well. A newcomer can find here all basic notions, terminology, and bases for the chosen mathematical models. A specialist will likely find an alternative viewpoint on the subject matter.

To simplify the reading of the book, a mathematical supplement is given in Appendices A.1 and A.2. One can find some elements of the theory of binary relations in Appendix A.1, and some elements of theory of real functions, including integration over finitely additive measure, in Appendix A.2.

The index contains both new terminology and that specific to and important for the reading of this book.

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<sup>2</sup> Another form of uncertainty aversion that induces a similar criterion, but for a convex set  $\mathbf{P}$ , was used later in [24].

As one can see from the above, a close connection exists among all the chapters of this book. At the same time, one can read each chapter separately. If one wishes only to become familiar with the statement of the problems, the results, and their interdependencies, it is enough to read the preliminary section to each chapter. Precise formulations and proofs are given in the following sections of each chapter. This peculiarity of the book was the result of experience in lecturing on such a multifaceted subject.



# Chapter 2

## Decision Systems

The correct statement of the laws of physics involves some very unfamiliar ideas which require advanced mathematics for their description. Therefore one needs a considerable amount of preparatory training even to learn what the words mean.

---

Richard Feynman

### 2.1 Preliminaries

Decision theory emerged from the requirements of diverse fields of human activity such as medicine, gambling, politics, warfare, economics and finance, and engineering. Perhaps this is the reason for the terminological diversity that sometimes impedes not only mutual understanding between specialists in different fields but also the development of decision theory itself. In this sense, control theory has been more fortunate, for its terminology turned out to be common to many spheres of its application.

There are two points of view on the relationship between decision theory and control theory. According to one such view, they have nothing in common. According to the other, these theories are gradually converging because the differences between them are not fundamental [14, 18]. The author of this book is an adherent of the latter viewpoint, and proposes the following motivation.

In control theory one studies a *control system* that consists of a pair of objects: a *plant* and a *controller*. In decision theory one studies a pair consisting of a *decision situation* and a *decision-maker*. It is natural to call such a pair a *decision system*. A *control system* is defined similarly.

The problem of choice of a decision or a control—an action that produces some consequence—is a problem common to both systems. In both systems, one may encounter two basic difficulties in the process of making this choice: dynamics and uncertainty. The development of control theory began in engineer-

ing, and the dynamics of plants became its central problem. The development of decision theory began in economics, and uncertainty became its central problem. While this dichotomy still exists, more and more attention is now being devoted to uncertainty in control theory [53] and to dynamics in decision theory [14].

But there is still an essential difference. Whereas the choice of decision criterion is at the center of decision theory, it is still on the periphery in control theory.

A systematic mathematical study of a control system becomes possible only if we define mathematical models of its components: the controlled plant, the controller, and the experiment (observation) the controller can perform over the plant. The same must be true about a decision system. Therefore, in this chapter we introduce the notion of a decision system and mathematical models of its components: the decision situation, the decision-maker, and the experiment (observation) the decision-maker can perform over the decision situation. An attempt to define a model of the second component of a decision system (the decision-maker) may seem surprising if we do not mention that our model concerns only the sequence of specific operations any decision-maker performs in the process of decision-making.

The existence of two types of real decision situations—nonparametric and parametric—leads to the existence of two types of model of these situations, called respectively the *lottery model* and the *matrix model*.

The *lottery model* has the form

$$S_l = (Z_l, I_l),$$

where

- $Z_l = (U, C, \psi(\cdot))$  is called a *lottery scheme*;
- $U$  is the set of possible decisions or actions;
- $C$  is the set of all possible consequences;
- $\psi : U \rightarrow 2^C$  is a model of the cause–effect mechanism of the decision situation in the form of a multivalued mapping;
- $I_l$  is some datum or *regularity* of this mechanism, available to the decision-maker by the moment at which the decision is made.

The *matrix model* has the form

$$S_m = (Z_m, I_m),$$

where

- $Z_m = (\Theta, U, C, g(\cdot, \cdot))$  is called a *matrix scheme*;
- $U$  and  $C$  have the same meaning as in the lottery model;
- $\Theta$  is the set of values of the parameter that affects a decision consequence;
- $g : \Theta \times U \rightarrow C$  is a model of the cause–effect mechanism of the decision situation in the form of a function of two variables;

- $I_m$  is some datum or *regularity* of this mechanism, available to the decision-maker by the moment at which the decision is made.

It turns out (Theorem 2.1) that the matrix and lottery models are equivalent, that is, each of them can be used to describe parametric as well as nonparametric situations.

In real parametric situations, two types of experiments (observations) are conceivable: first, observation of the parameter  $\theta \in \Theta$  if it is physically possible, and second, the observation of the consequence  $c \in C$  of the previous decision. In nonparametric situations a decision-maker can observe only the consequences  $c \in C$  of previous decisions. In real life, parametric situations are less frequent than nonparametric situations, but their natural (matrix) model happens to be more convenient for the purposes of accounting and processing the data obtained in the experiment.

So the model of experiment in a nonparametric situation has the form

$$h_l : C \rightarrow Y_{h_l},$$

while in a parametric situation it has the form

$$h_m^\theta : C \rightarrow Y_{h_m^\theta}$$

or

$$h_m^c : \Theta \rightarrow Y_{h_m^c},$$

where the  $Y$ 's are the sets in which the results of the experiments take values.

In describing a nonparametric situation in terms of the matrix model, the set  $\Theta$  of values of the parameter  $\theta$  is introduced artificially. That is why the parameter  $\theta$  may not be observed explicitly. But due to the equivalence of the matrix and lottery models—due to their one-to-one correspondence—the artificial parameter  $\theta$  characterizes the cause–effect mechanism of the nonparametric situation.<sup>1</sup>

We assume further that any decision-maker that happens to be in a situation demanding a decision has her own preference regarding the consequences of her decisions. This preference, or more precisely, *preference relation*<sup>2</sup> that the decision-maker establishes on the set  $C$  of consequences, is denoted by  $\beta_C$ . It turns out that the decision situation scheme  $Z$  and the preference relation  $\beta_C$  determine together whether there is uncertainty in the decision system. Theorem 2.2 establishes necessary and sufficient conditions for the existence of uncertainty in a decision system. In this way, uncertainty reveals itself as a system notion: there is no uncertainty without a definite decision-maker, without his

<sup>1</sup> Observation of consequences allows one to construct an estimate  $\hat{\theta}(y, u)$  of this parameter. This is discussed in Chapter 8.

<sup>2</sup> See Appendix A.1.

personal preference relation  $\beta_C$ . In two different decision systems that share the same decision situation but have different decision-makers (different preference relations  $\beta_C$ ), uncertainty can exist in one system and be absent in the other. It turns out that uncertainty in a decision system is what creates the problem of choice of decision criterion, or in other words, the problem of choice of a decision, a problem with a strong psychological accent.

## 2.2 The Structure of Decision Systems

Let us begin by considering some examples.

*Example 2.1.* In many folktales there appears the following type of episode: a hero, together with his beloved and his treasure, is riding his horse along a road and eventually comes to a crossroads, where there stands a stone with the following inscription:

If you go left you may lose your beloved  
 If you go right you may lose your horse  
 If you go straight you may lose your wealth

The hero cannot turn back. After all, he is the hero! He must choose one of the three roads on which to continue his travels. He is keenly aware of the possible consequences of each of the three decisions. Our hero, preferring his beloved and his horse to his wealth, with little deliberation goes straight. But he eventually reaches another crossroads, and another stone, with the following inscription:

If you go left you may lose your beloved or your wealth  
 If you go right you may lose your horse or your wealth  
 If you go straight you may lose your horse or your beloved

This choice is more difficult. Being a typical hero, he will most likely prefer his beloved to his horse and therefore turn to the right. After all, typical heroes value their beloved more than anything else. But eventually he is confronted with a new crossroads and a new stone:

If you go left you may lose your horse or your wealth or your beloved  
 If you go right you may lose your horse or your wealth or your beloved  
 If you go straight may lose your horse or your wealth or your beloved

And now the hero has a problem: wherever he goes, he is in danger of losing his beloved. His preferences (or priorities)—the beloved is more valuable than the horse and the horse is more valuable than wealth—do not help him to choose the optimal road. A conscious choice, a reasonable preference of one road over another, is impossible in this situation: the hero's preference for one or another

of the possible consequences of his choice do not apply at this crossroads. The hero can, of course, throw a die, thereby submitting himself to the whims of chance. But if he wishes to justify his decision, he needs some additional data as to which direction offers the least likelihood that he will lose his beloved and a good chance of losing nothing. In order to obtain such data he addresses an old raven sitting on a branch in a nearby oak who has seen many travelers at this crossroads. It turns out that indeed, the probabilities are different: on the left-hand road, there is a greater chance of losing nothing and less chance of keeping his beloved than on the right-hand road. Now the hero's decision depends on his personality: a self-confident hero—perhaps he views himself as an invincible fighter—may turn to the left, while the more cautious, more prudent hero turns right, for he is ready to lose his horse or his treasure if he can keep his beloved. Which of the two is our hero? Which type is more “rational”?

*Example 2.2.* A college graduate wants to choose a direction for further studies: humanities, sciences, or engineering. He may be more apt at one of the three, but he does not know which one. If his choice coincides with his abilities, his life will be more satisfying. But how is he to make a choice if he does not know his abilities? Again, like the hero in the previous example, our graduate must obtain further data about his state, where we assume that he is in one of only three possible states, that is, that he possesses the greatest ability in only one of the three areas. Our student may take some aptitude tests, but there are no tests that can rule out the possibility of a mistake. So the uncertainty of the situation cannot be eliminated.

*Example 2.3.* The expression “Hobson's choice” refers to a situation in which there is an appearance of choice, but in fact no choice at all. Thomas Hobson (1544–1610) was an entrepreneur in Cambridge, England, where he operated a livery stable with horses for hire. Having found that customers preferred the best horses, which were becoming overworked, Hobson instituted the rule that the customer had to accept the first horse in line at the stable gate. Thus the customer was unable to indulge any preference regarding the horse's color or temperament. Today we might say that by paying the fee to hire a horse, the customer has paid for the right to participate in a lottery with the horses as the prizes (or the consequences): the customer will receive one horse from the set of horses held in the stable. In fact, there is no choice (or decision-making) here. The choice has been offered to the client earlier: knowing the rules of the stable, he could agree to hire a horse or refuse to do so. Here we would say that he chooses between two alternatives:

- (1) participation in the lottery with “any of Hobson's horses” as a consequence;
- (2) participation in the lottery with “none of Hobson's horses” as a consequence.

Perhaps it was not so easy to refrain from participating in the Hobson's horse lottery. Different ethical (aesthetic, moral, social) considerations could favor the

decision to hire an arbitrary Hobson's horse. For example, it is possible that those who frequented the stables were members of a particular group, or a club, to which it was important to belong, and thus it was worthwhile to risk getting a bad horse.

Decision theory today considers only ethically neutral alternative acts. A lottery, in which the consequences are monetary gains and the preference relation regarding the consequences is determined by the gain size, is the traditional example of such neutrality. Let \$10 be the price for the right to participate in any of the two following lotteries: the first lottery offers the gains \$1,000,000, \$10, \$0, while the second offers \$10,000, \$100, and \$0. Which lottery do we prefer? Most likely, we will not be able to make an informed choice, for the situation is rather uncertain. Of course, the desire for riches might make some of us choose the first lottery. But such a choice would be opposed to the neutrality of the alternatives. Therefore, we would first wish to get some complementary data regarding the probabilities of gains in each of the two lotteries. Here we can say that the uncertainty in the choice of the best alternative appears due to the uncertainty of the situation: the same act can result in different consequences, and the same consequence can result from different acts.

One should not think that such an ambiguity in the consequences of our actions appears only in situations in which a human being deals with systems created by human beings, i.e., those that function not according to the laws of nature, but according to other laws devised by people. The influences of our actions on various physical or chemical systems and their responses (or the consequences of our actions) have been studied in control theory and its applications, a discipline close to decision theory and its applications. Ambiguity in the aforementioned responses also leads to decision-making under uncertainty. Here is an example.

*Example 2.4.* Consider the construction of some part or subassembly that will go into a certain technological device. During construction, the part will be tested by subjecting it to a certain influence (a temperature field, a vacuum, ionic/molecular bombardment, etc.) for a certain period of time  $t$ . Each part requires its own processing time  $t^*$ . But time  $t^*$  is unknown at the moment that processing time  $t$  is chosen. However, we know that  $t^* \in [T_1, T_2]$ , and in order for a part to be certified, it is necessary that time  $t$  not deviate from  $t^*$  by more than  $\Delta t$ , that is,  $t$  should be chosen in such a way that  $|t^* - t| < \Delta t$ . If this condition is not satisfied, the part is declared unfit. It is clear that if  $\Delta t < |T_1 - T_2|/2$ , there is no chance for all the parts to be certified. Thus, the problem is to choose a value for  $t$  that minimizes the quantity of unfit parts. So far, what is important for us is to choose a certain value of  $t$ . But when  $t^*$  is not known in advance, the result (or the consequence) of the operation is uncertain: for any choice of  $t$  at least two outcomes are possible: the part is good or it is bad. The set of possible

alternatives is the set of all possible time values  $t \in |T_1 - T_2|$ . So the question is, what value of  $t$  shall we choose?

This problem resembles the game “guess the number,” in which a player is to guess a number chosen from  $[1, \dots, 10]$  by another person. If he guesses correctly, he wins a prize, and if not, he loses something. This game is artificial, whereas in our technological situation we are dealing with the laws of physics and chemistry. Here ambiguity appears, due to the fact that the system is open to an external influence about which we know nothing. In this sense, there is no principal difference between physical and socioeconomic systems: both can be under the influence of factors “chosen by us” or “by the decision-maker” as well as “chosen by something (somebody) else,” and hence our decisions and corresponding actions do not determine the consequence uniquely.<sup>3</sup>

It is easy to see that all our examples have much in common. Indeed, everywhere in the above examples, someone—a decision-maker—happens to be in a situation demanding that he make a decision (alternative, action) out of a certain set of possible decisions. In the first example, it is the choice of the road; in the second, it is the choice of the educational profile; in the third, it is the choice of the lottery; in the fourth, it is the choice of the processing time. In all these examples, the choice—making a decision—implies an action leading to certain consequences. We call such situations *decision situations*.

In all our examples, at least one action may have several possible consequences, only one of which will take place. In all the examples, the consequence of interest to the decision-maker can be the result of more than one of his actions, and the decision-maker does not know for certain which action generates this consequence. It is for this reason that such actions are called “decisions under uncertainty.”

In all the examples, the decision-maker may have some data about “probabilities” in favor of one or another consequence. Sometimes, before making a decision, a decision-maker can execute some experiment (observation) that would provide additional data about these “probabilities.” Therefore, data about the decision situation are usually divided into a priori data, those that the decision-maker has before the experiment, and a posteriori data, those that he has after the experiment.

Note that in all our examples there is no dependency between consequences of different actions: whatever the consequence of an action, it in no way influences the consequences of other actions. We limit ourselves in this book to this independence of consequences. As we shall see later, this independence, induced by the structure of the cause–effect mechanism of a situation, increases in a certain sense the complexity of the decision situation.

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<sup>3</sup> This situation is similar to the situation in which at each moment of time, a portfolio manager or a trader makes a decision to change or not to change the structure of his investment portfolio.

Finally, note that our examples naturally form two groups. The first group includes Examples 2.1 and 2.3, while the second includes Example 2.2 and 2.4. Namely, in situations belonging to the second group, there is an explicitly present “physical”<sup>4</sup> parameter that along with the decision-maker’s actions influences the consequences. Thus in Example 2.2, this parameter is the true but unknown vocation of a college graduate; in Example 2.4, it is the true, but unknown to the decision-maker, processing time of a machine part. In situations of the first group, there is no such explicit parameter. Nevertheless, the sets of possible outcomes (consequences) that correspond to a decision-maker’s actions still are such that he does not know which of his actions will lead to the required outcome.

We call the situations of the first type (first group) *nonparametric situations*, while the situations of the second type (second group) are called *parametric situations*. In this connection there are two types of experiments, or observations, that may be provided by a decision-maker. In situations of nonparametric type, a decision-maker can observe only consequences (probably distorted) of his previous actions (decisions). So, in Example 2.1, the hero obtains such observations from the raven. In the situations of parametric type, the decision-maker also may observe consequences of his previous actions. But sometimes, before making a decision he may observe the value (which may be distorted) of the parameter that affects the consequences. So in Example 2.2, the graduate may observe the results of some special tests. Note also that in parametric situations, both types of observation may sometimes be provided.

Finally, we note that all our examples have the same structure, a sketch of which is presented in Figure 2.1:

In the figure, DM = decision-maker, DMS = decision-making situation. They are connected by channels 1 and 2. Through channel 1 the decision-maker receives the data  $S = (Z, I)$  about the situation before making a decision. Through channel 2 the decision-maker makes decisions  $u \in U$ , which result in consequences  $c \in C$  in channel 3. Channel 4 is an experiment of the first type; channel 5 is an experiment of the second type. Finally,  $\theta \in \Theta$  is the unknown parameter in the parametric decision situation.

We call the structure consisting of a decision-making situation and a decision-maker a *decision system*, in analogy to control theory, where the pair consisting of a controlled plant and a controller is called a control system. If an experiment is provided, we shall say that we are dealing with a *decision system with experiment*.

Our goal is to study decision systems. For such a study we need mathematical models of all parts of the decision system: the decision situation, the decision-maker, and the experiment. Decision systems with experiment will be studied in Chapters 6, 7, and 8.

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<sup>4</sup> We use the word “physical” in the widest possible sense, meaning “actually existing,” following Bellman [7].

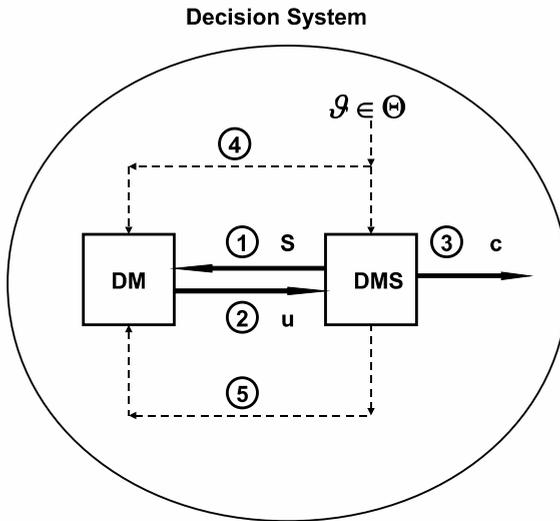


Fig. 2.1 The structure of a decision system.

## 2.3 Decision Situations

We suppose that everything that is unknown to the decision-maker is introduced into the decision system only by the decision situation. We shall construct a mathematical model of the decision situation in the form of the pair  $S = (Z, I)$ , where  $Z$  is the scheme of the decision situation and  $I$  is the data about the regularity of the cause–effect mechanism. The above-mentioned existence of two classes of decision situations—nonparametric and parametric—engenders two different models of decision situations.

### 2.3.1 The Scheme of a Decision Situation

For any situation we admit the following terminology and notation. We say that a decision-maker has to choose a decision  $u$  from some set  $U$  of all admissible decisions for a given situation, i.e.,  $u \in U$ .<sup>5</sup> We shall identify a decision  $u$  with an action that may generate some consequence  $c$  from the set of all possible consequences  $C_u$  for this decision.

<sup>5</sup> Henceforth, if it is not specified, we mean arbitrary sets.

Let

$$\bigcup_{u \in U} C_u = C$$

denote the set of all possible consequences of the situation.

We call the triple

$$Z_l = (U, C, \psi(\cdot)) \quad (2.1)$$

a *lottery scheme of a decision situation*, or simply a *lottery scheme* for short. The mapping  $\psi(\cdot)$  will sometimes be written in the form of a parametric set  $(C_u, u \in U)$ , and the lottery scheme analogously in the form of the triple

$$Z_l = (U, C, (C_u, u \in U)).$$

We call the quadruple

$$Z_m = (\Theta, U, C, g(\cdot, \cdot)) \quad (2.2)$$

a *matrix scheme*. Here  $\Theta$  is the set of all possible values of the unknown parameter  $\theta$  that may be chosen by someone other than a decision-maker.

In the lottery scheme (2.1), the multivalued mapping

$$\psi: U \rightarrow 2^C \in \{2^C\}^U$$

describes the cause–effect mechanism of generation, or generator, of the consequences in a decision situation. In the matrix scheme (2.2), the single-valued mapping

$$g: \Theta \times U \rightarrow C$$

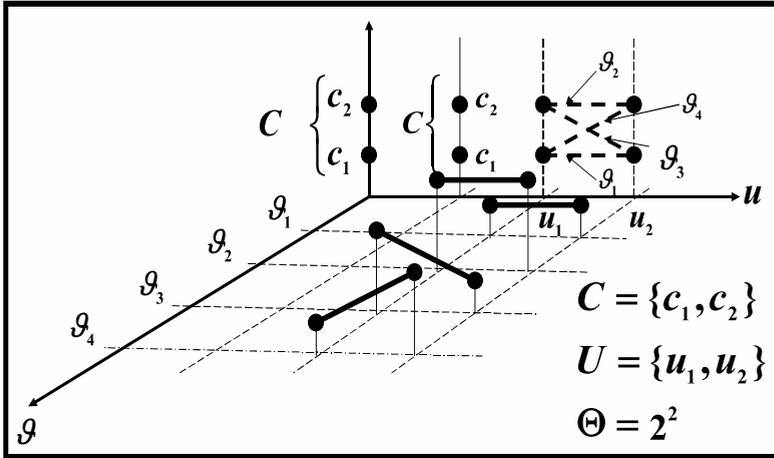
is another description of the cause–effect mechanism of generation of consequences.

If the decision-maker knows nothing about the regularity of the cause–effect mechanism, then the model of the situation is reduced to, or becomes exhausted by, its scheme.

It seems natural to use the lottery scheme  $Z_l$  in modeling a nonparametric situation, and the matrix scheme  $Z_m$  in modeling a parametric situation. However, it is not difficult to see that a parametric situation can be described in terms of a lottery scheme  $Z_l$ , and a nonparametric situation in terms of a matrix scheme  $Z_m$ .

Indeed, let the nonparametric situation be described in terms of the lottery scheme

$$Z_l = (U, C, \psi(\cdot)).$$



**Fig. 2.2** Consistency between a lottery and a matrix scheme.

In order to describe this situation in terms of the matrix scheme  $Z_m$ , it is necessary to construct the set  $\Theta$  and the mapping  $g(\cdot, \cdot)$  in such a way that will conserve the “complexity” of the original lottery scheme.

Having noted that in response to the chosen action  $u \in U$ , the cause–effect mechanism generates only one consequence  $c \in C_u$ , we describe the *complexity* of the original nonparametric situation in terms of the set of all complex events

$$\hat{\Theta} = \{ \hat{\theta} \in (U \rightarrow C) : \hat{\theta}(u) \in C_u, \forall u \in U \}. \tag{2.3}$$

Here  $\hat{\theta}(\cdot)$  is a single-valued function with the domain of definition  $U$  (see Figure 2.2).

In addition, it is reasonable to evaluate the complexity of the situation described in terms of the lottery scheme (2.1) by the capacity  $m$  of the set  $\hat{\Theta}$ , assuming  $m(\hat{\Theta}) = m(C^U)$ .

In the case of finite sets  $U$  and  $C$ , it is obvious that the complexity is given by the number

$$\text{Card}(\hat{\Theta}) = \text{Card}(C^U). \tag{2.4}$$

Setting

$$\hat{\theta}(u) = \hat{g}(\hat{\theta}, u), \quad (2.5)$$

we obtain a matrix scheme  $\hat{Z}_m = (\hat{\Theta}, U, C, \hat{g}(\cdot, \cdot))$  in which, in contrast to the matrix scheme  $Z_m = (\Theta, U, C, g(\cdot, \cdot))$  describing some parametric situation, the parameter  $\hat{\theta}$  is artificial and may not have physical sense. Following Savage [73], one may call this artificially constructed parameter  $\hat{\theta}$  a *state of nature*.

The procedure defined by the operations (2.3) and (2.5) is called *parameterization*. Parameterization defines the mapping

$$\tau : \mathbb{Z}_l = \{Z_l = (U, C, \psi(\cdot))\} \rightarrow \hat{\mathbb{Z}}_m \{\hat{Z}_m = (\hat{\Theta}, U, C, \hat{g}(\cdot, \cdot))\}, \quad (2.6)$$

where  $\mathbb{Z}_l$  and  $\hat{\mathbb{Z}}_m$  are classes of all lottery schemes and matrix schemes equivalent to them in capacity.

It is clear that the matrix scheme  $\hat{Z}_m$  can always serve to restore the original lottery scheme  $Z_l$ . Indeed, define the multivalued mapping  $\psi(\cdot)$  according to the following rule:

$$\psi(u) = \{\hat{g}(\hat{\theta}, u) : \hat{\theta} \in \hat{\Theta}\} \quad \forall u \in U. \quad (2.7)$$

We call this procedure *compression*. It defines the inverse mapping

$$\chi : \hat{\mathbb{Z}}_m = \{\hat{Z}_m = (\hat{\Theta}, U, C, \hat{g}(\cdot, \cdot))\} \rightarrow \mathbb{Z}_l = \{Z_l = (U, C, \psi(\cdot))\}. \quad (2.8)$$

The mappings (2.6) and (2.8) are bijective, and therefore the set  $\mathbb{Z}_l$  and  $\hat{\mathbb{Z}}_m$  are equivalent (have equal capacities).

Then taking into account (2.6) and (2.8), we have

$$\begin{aligned} \chi[\tau(Z_l)] &= \chi[\tau(U, C, \psi(\cdot))] \\ &= \chi[\{\theta \in (U \rightarrow C) : \theta(u) \in \psi(u), \forall u \in U\}, U, C, \{g(\cdot, \cdot) : \\ &\quad g(\theta, u) = \theta(u), \forall \theta \in \Theta, \forall u \in U\}] \\ &= (U, C, (\{\theta(u) : \theta \in \Theta\}, \forall u \in U)) = (U, C, \psi(\cdot)) = Z_l, \forall Z_l \in \mathbb{Z}_l. \end{aligned} \quad (2.9)$$

This result may be written as our next theorem.

**Theorem 2.1.** *The class of decision situations that can be represented by the matrix scheme  $Z_m$  coincides with the class of situations that can be represented by the lottery scheme  $Z_l$ , i.e.,*

$$\mathbb{Z}_m = \mathbb{Z}_l. \quad (2.10)$$

Thus we can use either of the two schemes  $Z_l$  and  $Z_m$  to describe any decision situation, parametric or nonparametric.

The majority of real-life decision situations seem to be of nonparametric type. However, in many real parametric situations the capacity of the set  $\Theta$  of possible values of the parameter  $\theta$  is less than the capacity of the set  $C^U$ : some of

the complex events  $\theta(u)$  may be absent. This makes the parametric situation less complex. So in our Example 2.2, the set  $\Theta$  (the set of all possible abilities of the graduate student) consists of three elements, whereas the set  $C^U$  consists of twenty-seven elements! In the degenerate case, in which it is known that the graduate is talented in only one type of activity (for example, engineering), the uncertainty of choice disappears; the complexity of the situation is minimal (zero?).

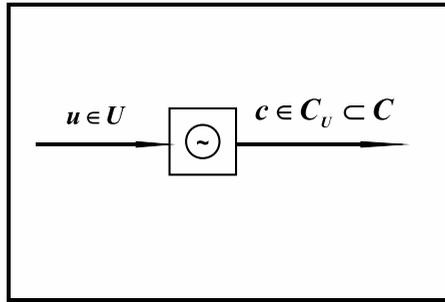
It is clear that when  $m(\Theta) \leq m(C^U)$ , one should describe the parametric decision situation in terms of the matrix scheme. In other words, compression of this scheme into the lottery scheme can lead only to an unjustified complication of the situation model, or to the loss of data about the simplicity of the parametric situation. Indeed, if the real set  $\Theta$  of the parametric situation is preserved (is not forgotten) in the operation of compression, then the lottery scheme will consist of the matrix scheme completed by the useless multivalued mapping. If, on the other hand, after compression  $\chi$  one does not preserve, or forgets, the data about the set  $\Theta$ , then such a compression turns out to be a surjection of the set  $\mathbb{Z}_m$  onto the set  $\mathbb{Z}_l$ : a single lottery scheme (the image) corresponds to more than one matrix scheme (the preimage).

### 2.3.2 Data about the Unknown

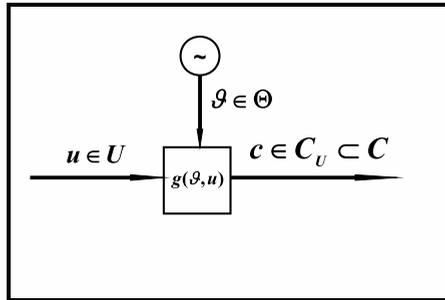
Looking back at the examples of decision situations from Section 2.2 and fitting them to the decision schemes  $Z_l$  and  $Z_m$ , one can see that a scheme by itself is insufficient to model a situation. In Examples 2.1 and 2.2, not only does the decision-maker know the schemes  $Z_l$  and  $Z_m$ , but he receives as well some additional data about peculiarities of the appearance of consequences of every decision. We call these data the *data about the unknown*.

When we say “unknown,” we mean that the decision-maker does not know in advance which consequence  $c \in C_u$  will be the result of the chosen action  $u \in U$ . The notion of the unknown, with all its richness, can scarcely be formulated in precise mathematical language. In this book, we shall confine our treatment of the unknown to following, to a certain extent, R. Bellman [7]. Namely, we shall assume that the consequence  $c \in C_u$  is induced by some cause–effect mechanism, or generator of consequences, which is switched on by the chosen action  $u \in U$  in a way shown in Figure 2.3 for nonparametric situations and in Figure 2.4 for parametric situations. It is precisely this cause–effect mechanism that represents in our model the dependence, specific to each decision situation, of consequences on actions. Clearly, the range of these dependencies is very wide, from functional dependence to complete independence.

It is natural to suppose that the source of unknowns is the actual environment, or the reality of the physical—in the widest possible sense of this word—world,



**Fig. 2.3** A nonparametric situation.



**Fig. 2.4** A parametric situation.

in which the decision situation takes place. So, for example, the profit from operations on the stock exchange depends on the totality of financial markets; the outcome of a disease depends on the entire state of medicine; the length of a human life depends on social conditions, but of course not on that only.

It is natural to consider the unknown consequence  $c \in C_u$  as random, and the cause-effect mechanism as the generator of such randomness. The specificity of a decision situation consists then in some regularity of the considered random phenomenon. Since these regularities are proposed by modern probability theory, it would seem that there should be no new difficulties. But this is not the case. Already Borel in [4] remarks that the world of random phenomena is much wider than that of its parts, which is described in terms of probability the-

ory and remarks on the absence of scientific means of studying this world. In [47], Kolmogorov says in this respect the following: “Speaking of randomness in the usual sense of this word, we mean those phenomena in which we do not find regularities, allowing us to predict their behavior. Generally speaking, there are no reasons to assume that phenomena random in this sense are subjected to some probabilistic laws. Hence, it is necessary to distinguish the randomness in this *broad sense* and *stochastic* randomness (which is the subject of probability theory).”

However, what do the words “we do not find regularities, allowing us to predict their behavior” mean? We should scarcely understand them in the sense that such regularities do not exist, that in the range between the determinism of functional dependence and complete uncertainty there is only the regularity of stochastic randomness. More likely, they express the need to find the regularities of nonstochastic random phenomena. We shall return to this topic later.

So far, independent of the form—perhaps even a nonmathematical one—in which the regularity of the cause–effect mechanism is expressed, we shall denote it by the symbol  $I$ . Denote the lottery and the matrix models of a decision-making situation by  $S_l = (Z_l, I_l)$  and  $S_m = (Z_m, I_m)$  respectively.

We have already seen that the matrix and lottery schemes,  $Z_m$  and  $Z_l$ , are equivalent ways of describing decision situations: each of them can be used for the description of a parametric as well as a nonparametric situation. It seems that the models  $S_l$  and  $S_m$  are equivalent in this sense as well: making use of the operations of parameterization  $\tau$  and compression  $\chi$ , the regularity  $I_m$  could be rewritten in terms of  $I_l$  and vice versa. However, so far one can see this only for a concretely given regularity  $I$ .

In particular, let us demonstrate this equivalence of the models  $S_m$  and  $S_l$  for situations with finite sets of decisions  $U$  and consequences  $C$  in the case of stochastic regularity of the cause–effect mechanism.

For a certain situation, let its lottery model be  $S_l = (Z_l, I_l)$ , where

$$\begin{aligned} Z_l &= (U, C, \{\overline{C_k}, k = \overline{1, d}\}), \\ U &= (u_1, u_2, \dots, u_d), \\ C &= (c_1, c_2, \dots, c_t) = \bigcup_{k=1}^d C_k, \end{aligned}$$

and the stochastic regularity  $I_l$  has the form of a family of probability distributions

$$I_l = Q = \{Q_k, k = \overline{1, d}\}, \quad Q_k = (q_k(c_1), q_k(c_2), \dots, q_k(c_t)).$$

The matrix scheme  $Z_m$ , constructed on the basis of the lottery scheme  $Z_l$  according to the parameterization  $\tau$  (2.6), has the form  $Z_m = (U, C, \hat{\Theta}, g(\hat{\cdot}, \cdot))$ , where the artificial parameter  $\hat{\Theta}$  takes values in the set  $\hat{\Theta}$ ,  $\text{Card}(\hat{\Theta}) = n = t^d$ ,

i.e.,

$$\begin{aligned}\hat{\Theta} &= (\hat{\theta}_1, \hat{\theta}_2, \dots, \hat{\theta}_n), \quad \hat{\theta}_\mu : U \xrightarrow{\mu} C, \\ \mu &= \overline{1, n}, \quad \hat{\theta}_\mu(u_k) = g(\hat{\theta}_\mu, u_k), \quad \forall k = \overline{1, d}.\end{aligned}$$

Here  $\hat{\theta}_\mu$  is a “complex” event, consisting of the “simple” events  $c_{v\mu k}$ ,  $v = \overline{1, t}$  that constitute  $\hat{\theta}_\mu$ .

In order not to complicate our example, suppose that there is no dependency between the consequences of different actions, in other words, that consequences are overall independent. Then the regularity  $I_m$  in terms of the probability distribution over the set  $\hat{\Theta}$  is given as

$$I_m = P(\hat{\Theta}) = (p(\hat{\theta}_1), p(\hat{\theta}_2), \dots, p(\hat{\theta}_n)), \quad (2.11)$$

where

$$p(\hat{\theta}_\mu) = \prod_{k=1}^d q_k(c_{v\mu}), \quad \mu = \overline{1, n}. \quad (2.12)$$

The matrix model  $S_m = (Z_m, I_m)$  obtained is equivalent to the original (primitive) lottery model  $S_l = (Z_l, I_l)$ . Indeed, we make the inverse transformation from  $S_m$  to  $S_l$ , the compression (2.8). To do this we need to reconstruct  $I_l$  from  $I_m$  in the form of the family

$$Q'_k = (q'_k(c_1), q'_k(c_2), \dots, q'_k(c_t)), \quad k = \overline{1, d}, \quad (2.13)$$

where

$$q'_k(c_v) = p\{\hat{\theta}_\mu, \mu = \overline{1, n} : g(\cdot, u_k) = c_v\}, \quad v = \overline{1, t}. \quad (2.14)$$

The model  $S_l = (U, C, \{C_k, Q'_k, k = \overline{1, d}\})$  constructed in this way coincides with the original model  $S_l$ .<sup>6</sup>

To prove this equivalence in the case of arbitrary sets  $C$  and  $U$  is technically more complicated. Nevertheless, in what follows we shall everywhere, unless otherwise specified, write down the model of the decision-making situation omitting the indices, i.e., as  $S = (Z, I)$ .

<sup>6</sup> For situations with stochastic regularity in the case of an arbitrary set  $C$  and finite set  $U$ , the equivalence of models is proved in [42]. In Chapter 6 we show the equivalence of the matrix and lottery models for nonstochastic regularity.

## 2.4 Experiments in Decision Systems

The models of experiment in decision systems discussed in this section will be used only in the last three chapters of this book. Therefore, the impatient reader may skip this section and proceed to the following one, returning here only before reading those chapters.

Suppose the decision-maker is in the situation  $S = (Z, I)$ . His knowledge about the regularity of the cause–effect mechanism may be enriched by means of an observation, or an experiment, over the decision situation.<sup>7</sup> A description of the regularity before observation is called *a priori*. We have already denoted it by the symbol  $I$ . The regularity enriched by the observation is called *a posteriori*, and we denote it by the symbol  $I_h$ . Suppose that  $I$  and  $I_h$  are linked by the relation

$$I_h = f(I, w_h), \quad w_h \in W_h, \quad (2.15)$$

where  $f$  is some algorithm of transition from the a priori description of the regularity  $I$  to the a posteriori description  $I_h$  under the observation  $w_h$  from some set  $W_h$ .

To limit the definition of experiment and make it more precise, we return again to our examples in Section 2.2. Obviously there are two types of experiments conducted by the decision-maker. An experiment of the first type is conducted by the hero in Example 2.1: the raven informs the hero of the consequences that followed the decisions of the hero’s predecessors. An experiment of the second type is conducted in Example 2.2 when the young man explores his inclinations toward different professional occupations.

Generalizing what we have said, we define the model of the experiment of the first type as the mapping

$$h_l : C \rightarrow W_{h_l}. \quad (2.16)$$

This corresponds to the structure of the decision system in Figure 2.5.

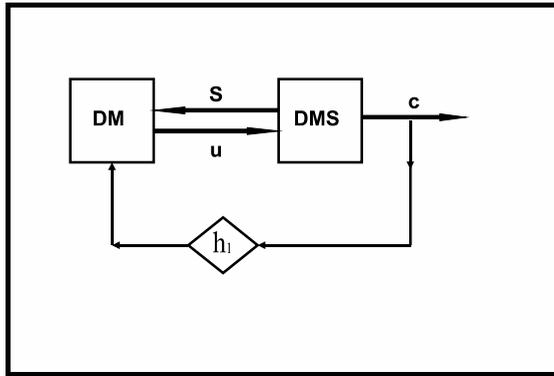
We define the model of an experiment of the second type as the mapping

$$h_m : \Theta \rightarrow W_{h_m} \quad (2.17)$$

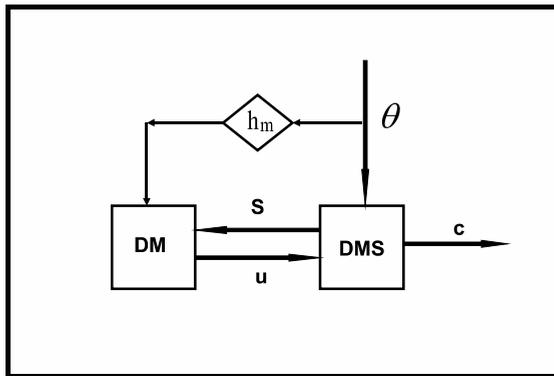
corresponding to the structure of the decision system in Figure 2.6. These figures make it clear that an experiment of the first type may be provided both in parametric and in nonparametric situations. An experiment of the second type may be provided only in parametric situations and only if the parameter  $\theta$  can be observed. Statistical studies consider primarily this type of experiment [14].

There is a profound difference between the two types of experiment. While observation  $w_m$  depends only on the parameter  $\theta \in \Theta$ , observation  $w_l$  is more complicated. According to (2.16), the observation  $w_l$  depends on the consequence  $c \in C$ , and hence depends on the decision  $u \in U$  that resulted in this

<sup>7</sup> We shall use the words “observation” and “experiment” as synonyms.



**Fig. 2.5** The structure of a decision system with an experiment of the first type.



**Fig. 2.6** The structure of a decision system with an experiment of the second type.

consequence. The decision-maker uses this observation in order to construct an a posteriori description of the regularity. Then this description is used for future decision-making. Thus a dependence arises between sequential decisions. Such dependencies produce some “information dynamics,” particularly important in multistep decision systems, i.e., those in which the decision-maker makes se-

quential decisions, remaining in the same situation and accumulating, step by step, data about the regularity of the cause–effect mechanism.

Note finally that refusing to consider random functions of  $\theta$  in our model of experiments in no way restricts the generality of this model. Indeed, let, for example, an experiment be written in the form of a random function of  $\theta$ , i.e., as

$$h_m : \Theta \times \Omega \rightarrow W_m, \quad (2.18)$$

where  $\Omega$  is the set of primitive events. This model can always be reduced to the model (2.17) by an obvious change of variables: it is enough to consider the direct product  $\Theta \times \Omega$  as a new set  $\Theta'$  of the unknown parameter.

## 2.5 The Decision-Maker

It follows from the examples of Section 2.2 that every decision-maker that happens to be in a decision situation has the goal of achieving what is in his opinion the best outcome or consequence. To fulfill this task, the decision-maker, equipped with the mathematical model of the situation  $S = (Z, I)$ , has to choose the action generating this outcome. We call this choice the *decision problem* or the *problem of choice*. To solve this problem, the decision-maker must perform, roughly speaking, the following sequence of steps:

1. Establish his personal preference relation<sup>8</sup> (or in other words, a criterion) on the set of all possible outcomes  $C$ . Denote this preference relation by  $\beta_C$ .
2. Determine the best, according to  $\beta_C$ , consequence  $c^0 \in C$ .
3. Establish some preference relation (criterion) on the set of all admissible actions  $U$ . Denote this preference relation by  $\beta_U$ .
4. Determine the best, according to  $\beta_U$ , action  $u^0 \in U$ .

These four steps constitute a *compound decision problem*, compound because each step in this sequence is a certain mathematical problem. Even the first step—establishing a personal preference relation  $\beta_C$ —is a mathematical problem: this relation must satisfy certain conditions if one wants to remain on stable ground.

The second and the fourth steps—the search for  $c^0$  and the search for  $u^0$ —are optimization problems that, generally speaking, can be solved by appropriate mathematical methods. These two steps are not problems specific to decision-making only. The problem specific to decision-making is that of the third step, namely, the problem of construction of the preference relation  $\beta_U$ . This problem is the kernel of the compound decision problem, and therefore, in speaking of such problems we sometimes will mean only the problem of the third step.

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<sup>8</sup> See Appendix A.1.

It is natural here to note that the data on the regularity  $I$  of the situation can only simplify the decision problem. Therefore, following our model of the decision situation, we say that if we know nothing of the situation but the scheme  $Z$ , then we limit the model  $S$  of the situation to its scheme, i.e.,  $S = Z$ . Nevertheless, the situation cannot be considered separately from the decision system, or more accurately, independently from  $\beta_C$ . Is it not obvious that some situation can be simple for one decision-maker, i.e., for his preference relation  $\beta_C$ , and be complicated for another decision-maker that has another preference relation  $\beta_C$ ? Suppose, for example, that the first is a colorblind person in a situation in which the consequences are colors.

But the model  $S$  (or just the scheme  $Z$ ) of the situation and the preference relation  $\beta_C$  do not define the unique preference relation  $\beta_U$ . Indeed, it was already in Example 2.1, at the third crossroads, where we saw that two different heroes in the same situation, having the same preference  $\beta_C$ , later chose two different preferences  $\beta_U$  and correspondingly made two different decisions, i.e., chose two different roads. Each one justifies his choice in his own way, and each one is right—in his own way. But then we are tempted to classify these choices as arbitrary! To eliminate such arbitrariness it is necessary to transform the third step into a precise mathematical problem.

Before doing this, let us turn again to our examples. In particular, in Example 2.1, at the first crossroads, the hero of the story, having his preference  $\beta_C$  with respect to consequences, makes a decision without any speculation. In this decision situation, the choice of the road is uniquely determined by the hero's preference  $\beta_C$ , because a single consequence corresponds to only one road. We generalize this episode in the following way. Consider the situation in which the mapping  $\psi(\cdot)$  of the decision set  $U$  in the set  $C$  of consequences is single-valued. Denote such a mapping by  $\hat{\psi}$ . The inverse mapping  $\hat{\psi}^{-1} : C \rightarrow U$  is not necessarily single-valued, and the consequence  $c \in C$  can be generated by any element of some set  $U_{c \in C} \subset U$ . Note that the sets  $U_{c_i}$  and  $U_{c_j}$  are disjoint for all  $c_i, c_j, i \neq j$ . Thus the mapping  $\hat{\psi}^{-1}$  decomposes the decision set  $U$  into a system  $\hat{U}$  of disjoint subsets  $\{U_c, c \in C\} = \hat{U}$  in such a way that  $\hat{\psi}^{-1}(c_i)\beta_C\hat{\psi}^{-1}(c_j)$  follows from  $c_i\beta_C c_j$ . Thus if the preference  $\beta_C$  is satisfied for the set of images  $C$ , then the same  $\beta_C$  is satisfied for the set of preimages  $\hat{U}$ . But we denoted the preference relation on the set  $U$  by the symbol  $\beta_U$ . Thus the mapping  $\hat{\psi}^{-1}$  *transports*, or *projects*, the preference relation from the set  $C$  onto the set  $U$  in such a way that if  $c_i\beta_C c_j$  and  $u_{c_i} \in U_{c_i}, u_{c_j} \in U_{c_j}$ , then  $\hat{\psi}(u_{c_i})\beta_C\hat{\psi}(u_{c_j})$ . Clearly, there exists a unique preference relation  $\beta_U$  that does not contradict the preference relation  $\beta_C$ , or to put it briefly, retains (or supports)  $\beta_C$ . We say that the third step of such a decision problem is *simple* or *degenerate* simply because there exists only one such preference relation  $\beta_U$ . Indeed, in this case one must solve only the optimization problem of the fourth step. In other words, the whole compound decision problem degenerates into the optimization problem.

So perhaps the decision problem does not become degenerate in the case that the mapping  $\psi(\cdot)$  is multivalued, and we do not know what the result of our action will be. That is frequently the case, but not always. Here is an example.

Let  $C_{u_1} = \{c_1, c_2\}$  and  $C_{u_2} = \{c_3, c_4\}$ , and according to  $\beta_C$  let  $c_3\beta_C c_1$ ,  $c_3\beta_C c_2$ ,  $c_4\beta_C c_1$ ,  $c_4\beta_C c_2$ , i.e., all results of the action  $u_2$  are preferred to all results of the action  $u_1$ . So we have here the multivalued function  $\psi(\cdot)$ . But there exists a unique  $\beta_U$  that does not contradict  $\beta_C$ . This  $\beta_U$  is produced by a specific combination of the inverse mapping  $\psi^{-1}(\cdot)$  (which is a surjection) and the preference relation  $\beta_C$ .

But the crossroads in the second and the third episodes from Example 2.1 are different. Here, at every crossroads, it is not known which of the several consequences will occur. In terms of the lottery model, in this situation the mapping  $\psi(\cdot)$  is many-valued, and it is such that the inverse mapping  $\psi^{-1}(\cdot)$  does not exist. Therefore, the hero's preference  $\beta_C$  does not determine the unique preference  $\beta_U$ . We say that this is the effect of uncertainty inherent in the situation.

And nevertheless, our hero is making his decision according to his personal, i.e., arbitrary, preference relation  $\beta_U$ .

We now resume our attempt to model the third step of the compound decision problem. Denote by  $\mathbb{S}$  the set of decision situations. For any decision situation  $S \in \mathbb{S}$  let  $\mathbb{B}_C$  denote the set of all preference relations  $\beta_C$  on  $C$ ,  $B_C$  the subset of  $\mathbb{B}_C$  available to the given decision-maker,  $\mathbb{B}_U$  the set of all preference relations  $\beta_U$  on  $U$ , and  $B_U$  the subset of  $\mathbb{B}_U$  available for the decision-maker.

**Definition 2.1.** We call the mapping

$$\pi : \mathbb{S} \times B_C \rightarrow B_U \quad (2.19)$$

a *projector* or *criterion choice rule*.

Denote by  $\Pi$  the set of all possible projectors or criterion choice rules  $\pi$ . Then it follows from the above that any decision-maker in a decision system reacts in his own way to the information about what is unknown, executing in his own way the operation of projection. In other words, any decision-maker in the decision system is a certain projector  $\pi$ , or equivalently, has his own criterion choice rule. This allows us to represent a decision-maker as a triple

$$\Phi = (B_C, B_U, \pi), \quad B_C \subseteq \mathbb{B}_C, \quad B_U \subseteq \mathbb{B}_U, \quad \text{with } \pi \subseteq \Pi. \quad (2.20)$$

This model allows us to describe the aforementioned arbitrariness in the following way: in the same situation  $S$ , with the same preference relations  $\beta_C$ , two different decision-makers  $\pi_1$  and  $\pi_2$  can establish two different preference relations  $\beta_U^1$  and  $\beta_U^2$  that can define two different best actions. A natural question is then, which one of them has chosen the best action? And what does it mean, "the best," if each decision-maker,  $\pi_1$  or  $\pi_2$ , has his own opinion, his own fourth

step “optimization” problem according to which each of them determines his own “best”?

The only exit from this arbitrariness is, most likely, to find and to put in the same group all decision-makers that have the same criterion choice rule. This will eliminate the arbitrariness inside the chosen group, or class: in the same decision situation  $S$ , all representatives of this class, having the same preference relation  $\beta_C$ , will establish the same preference relation  $\beta_U$ , and finally will make the same decision. However, is there such a criterion choice rule? And if it exists, then what kind of criterion  $\beta_U$  does it generate?

We are not ready to answer these questions. The problem of choice arises, as we saw, in decision systems with uncertainty, but we still do not have a precise notion of what it means for uncertainty to exist in a decision system.

## 2.6 Existence of Uncertainty in Decision Systems

We have already mentioned a few times the word “uncertainty,” but we have thus far never tried to make it precise. The sense of this word in everyday speech does not need any refinement. However, in everyday speech the word “probably” also does not need to be refined. Nevertheless, the quantitative characterization of the probable event demands a precise mathematical method, namely probability theory. Similarly, the notion of uncertainty demands at least a more precise definition. To begin with, note that in decision theory one has to be able to divide the whole class of decision problems into two subclasses: decision problems with and without uncertainty. For such a classification we need a criterion of existence of uncertainty in a decision system. Before we introduce such a criterion, we shall make some remarks.

There are two parts of a decision system that can be sources of uncertainty: the decision-maker and the decision situation. We suppose that the decision-maker can reveal uncertainty only in the choice of the preference relation  $\beta_C$ . So let us restrict decision systems to those for which any decision-maker can have a single personal preference relation  $\beta_C$  for the given situation. In this case, when  $\beta_C$  is fixed, the decision situation becomes the only source of uncertainty in the decision system, and thus in the corresponding decision problem.

Note that the data  $I$  can only decrease the uncertainty of the situation. Under these assumptions, uncertainty in the decision system comes only from the scheme  $Z$  of the situation.

To simplify our reasoning we limit the set  $\mathbb{B}_C$  to linear preferences (linear ordering) and define

$$\beta_C = (C, \succeq).$$

Let

$$Z = Z_l = (C, U, \psi(\cdot)).$$

Set

$$\mathfrak{K} = (2^C)^U,$$

that is, the set of all mappings  $\psi : U \rightarrow 2^C$ .

Denote the class of mappings that bring uncertainty into the decision system by  $K \subseteq \mathfrak{K}$ .

**Definition 2.2.** We say that decision  $u_1$  *dominates* decision  $u_2$  relative to  $\beta_C = (C, \succeq)$  if  $C_{u_1} \succ C_{u_2}$ , i.e.,  $u_1 \succ u_2$  if  $c_1 \succeq c_2 \forall c_1 \in C_{u_1}$  and  $\forall c_2 \in C_{u_2}$ , we have  $\text{Card}(C_{u_1} \cap C_{u_2}) \leq 1, C_{u_1} \neq C_{u_2}$ .

**Definition 2.3.** The procedure of constructing the preference relation  $\beta_U = (U, \succeq)$  in accordance with the following conditions is called *projecting*:

**Condition 1**

$$C_{u_1} \succ C_{u_2} \Rightarrow u_1 \succ u_2, \quad \forall u_1, u_2 \in U;$$

**Condition 2**

$$(C_{u_1} = C_{u_2}, \text{Card}(C_{u_1}) = 1) \Rightarrow u_1 \sim u_2, \quad \forall u_1, u_2 \in U.$$

**Definition 2.4.** A decision system contains uncertainty if the projecting procedure is not single-valued.

Obviously the multivaluedness of the mapping  $\psi(\cdot)$  is a necessary condition for the existence of uncertainty, i.e.,  $K \neq (2^C)^U$ . But it is clear that this condition is not sufficient, i.e.,  $K \neq (2^C)^U \setminus C^U$ .

Intuitively, one can easily propose some version of a sufficient condition. Suppose for simplicity that the sets  $U$  and  $C$  are finite, and the mapping  $\psi$  is single-valued everywhere but on one action  $u^* \in U$ . At this point, let the set of consequences be given by  $C_{u^*} = \{c_1, c_2\}$ ,  $c_1 \prec c_2$ . Suppose there exists the action  $u^{**}$  with the consequence  $C_{u^{**}} = \{c_3\}$  such that  $c_1 \prec c_3 \prec c_2$ . Then there exist two variants of the preference relation  $\beta_U$ : the first  $\beta_U^{(1)}$  with  $u^* \succ u^{**}$ , and the second  $\beta_U^{(2)}$  with  $u^* \prec u^{**}$ . So a sufficient condition can be formulated as follows: uncertainty exists if there is a situation  $S = Z$  and if there is a pair of decisions  $u^*, u^{**}$  with consequences  $C_{u^*} = \{c_1, c_2\}$  and  $C_{u^{**}} = \{c_3\}$  such that  $c_1 \prec c_3 \prec c_2$ .

But it turns out that one can prove a much stronger statement. In order to define the class  $K$  we need to prove our next lemma.

**Lemma 2.1.** We have  $\psi(\cdot) \in K$  if there exist a preference relation  $(C, \succeq)$  and distinct  $u_1, u_2 \in U$  and consequences  $c_1, c_2 \in \psi(u_1), c_3, c_4 \in \psi(u_2)$  such that  $c_1 \prec c_3, c_2 \succ c_4$ .

*Proof. Necessity.* Let some relation  $(C, \succeq)$  generate different projections on the set  $U$ . Assume the converse. This means that for all  $u_1, u_2 \in U, u_1 \neq u_2$ , precisely one of the following conditions holds:

$$\begin{aligned} \psi(u_1) \prec \psi(u_2), & \quad \text{Card}(\psi(u_1) \cap \psi(u_2)) \leq 1, \\ \psi(u_1) \succ \psi(u_2), & \quad \text{Card}(\psi(u_1) \cap \psi(u_2)) \leq 1, \\ \psi(u_1) = \psi(u_2), & \quad \text{Card}(\psi(u_1)) = 1. \end{aligned}$$

Then for the chosen  $(C, \succ)$ , according to Conditions 1 and 2, there exists a unique  $(U, \succ)$ , namely  $u_1 \succ u_2 \iff \psi(u_1) \succ \psi(u_2)$ , for all  $u_1, u_2 \in U$ . This contradicts the assumption that the relation  $(C, \succeq)$  generates different projections on the set  $U$ . In other words, this contradicts the ambiguity of the choice of the preference relations on the set of decisions  $U$ .

*Sufficiency.* Let the preference relation  $(C, \succ)$  satisfy the conditions of Lemma 2.1. Then there exist  $u', u'' \in U, u' \neq u'', c_1, c_2 \in \psi(u'), c_3, c_4 \in \psi(u'')$  such that  $c_1 \prec c_3, c_2 \succ c_4$ . Let  $(U, \succeq')$  be some preference relation on  $U$  (for example it may be a lexicographic relation with respect to  $(C, \succeq)$ ). Then one can take another preference relation  $(U, \succeq'')$  such that

$$u_1 \succeq'' u_2 \iff \begin{cases} u_1 \succeq' u_2, \\ u_1 \preceq' u_2, u_1 \preceq u_2. \end{cases}$$

The lemma is proved.

Next we need the following lemma.

**Lemma 2.2.** *The necessary and sufficient conditions are equivalent to the following conditions: there must exist distinct  $u_1, u_2 \in U$  and consequences  $c_1, c_2 \in \psi(u_1), c_3, c_4 \in \psi(u_2)$  such that either  $c_1 \prec c_3 \prec c_2$  or  $c_1 = c_3, c_2 = c_4, c_1 \neq c_2$ .*

*Proof. Necessity.* Suppose the conditions of Lemma 2.1 are satisfied for  $u_1, u_2, c_1, c_2, c_3, c_4$ . Then without loss of generality, one may assume that  $c_1 \preceq c_2$ . If  $c_1 = c_2$ , then setting  $c_1' = c_4, c_3' = c_4' = c_1, c_2' = c_3$ , we obtain for  $c_1', c_2', c_3', c_4'$  the first condition of Lemma 2.2. If  $c_1 \prec c_2$ , then when  $c_3 \succ c_2$ , for  $c_1' = c_4, c_2' = c_3, c_3' = c_2, c_4' = c_1$  we obtain again the first condition of Lemma 2.2. The same holds if  $c_3 \prec c_2$ . In the case  $c_3 = c_2$ , if  $c_1 = c_4$  we have at  $c_1' = c_1, c_2' = c_2, c_3' = c_3, c_4' = c_4$  the second condition of Lemma 2.2. And finally, if  $c_1 \neq c_4$ , for example at  $c_4 \succ c_1$ , then at  $c_1' = c_1, c_2' = c_2, c_3' = c_4, c_4' = c_3$  we have the second condition of Lemma 2.2

*Sufficiency.* Let the conditions of Lemma 2.2 be used for  $u_1, u_2, c_1, c_2, c_3, c_4$ . Then if  $c_1 \prec c_3 \prec c_4$ , the condition of Lemma 2.1 holds for  $c_1' = c_1, c_2' = c_2, c_3' = c_4', c_4' = c_3$ , and if  $c_1 = c_3 \succ c_2 = c_4$ , then the condition of Lemma 2.1 holds for  $c_1' = c_2, c_2' = c_1, c_3' = c_3, c_4' = c_4$ . The lemma is proved.

From these two lemmas we obtain the following theorem.

**Theorem 2.2.** *A decision scheme  $Z_l$  contains uncertainty if there is  $\psi(\cdot)$  such that one can determine distinct  $u_1, u_2 \in \text{Dom}(\psi)$  and  $c_1, c_2$  such that either  $\psi(u_1) = \psi(u_2) = \{c_1, c_2\}$  or there is also  $c_3$  different from  $c_1$  and  $c_2$  such that  $c_1, c_2 \in \psi(u_1)$  and  $c_3 \in \psi(u_2)$ .*

Now let  $Z = Z_m = (\Theta, U, C, g(\cdot, \cdot))$ , and this  $Z_m$  is equivalent to  $Z_l$  in the sense of Theorem 2.1. Then Theorem 2.2 can be reformulated for  $Z_m$  as follows.

**Theorem 2.3.** *A decision scheme  $Z_m$  contains uncertainty if and only if there is  $g(\cdot, \cdot)$  such that there exist distinct  $u_1, u_2 \in U$  for which either  $g(\Theta, u_1) = g(\Theta, u_2)$  and  $\text{Card}(g(\Theta, u_1)) = 2$  or there are  $\theta_1, \theta_2 \in \Theta$  such that  $g(\theta_1, u_1) \neq g(\theta_2, u_1) \neq g(\theta_1, u_2) = g(\theta_2, u_2)$ .*

In other words, these conditions define the class  $K$  of such schemes  $Z$  for which there exists  $\beta_C \in \mathfrak{B}_C$  such that one can project the pair  $(\beta_C, Z)$  into more than one preference relation  $\beta_U$ , i.e., more than one criterion of choice of the best action.

## 2.7 Criterion Choice Rule

Now we can return to the central, third, step of the compound decision problem from Section 2.5, namely to the construction of the preference relation  $\beta_U$ , or in other words, of the criterion of choice of the best decision. We already know that this problem becomes nontrivial if the decision situation contains uncertainty. We have already seen that in any decision system uncertainty exists if for a given preference relation  $\beta_C$  there is a scheme  $Z$  of the decision situation such that there exists more than one preference relation  $\beta_U$ , i.e., there is more than one criterion of choice of optimal decision. In other words, in this case the mapping (2.19) is multivalued, and the decision-maker can choose one of several criteria following his personal tastes. As is known [58], the complement of the decision scheme  $Z$  by the regularity  $I$  of the cause–effect mechanism does not, in general, remove uncertainty and the resulting arbitrariness from the decision system. To remove this arbitrariness one has to subordinate the projector  $\pi$  to some conditions. In some sense, these conditions, or axioms, may be considered as an axiomatic description, or a model, of the decision-maker. On the other hand, these conditions define a class of projectors, or decision-makers, that in the same decision situation  $S$ , having the same preference relation  $\beta_C$ , project the pair  $(S, \beta_C)$  in their common preference relation  $\beta_U$ . Thus at the fourth step of the compound decision problem, they will choose the same optimal decision.

To proceed further with this task it is necessary to make the set  $B_C$  and the set  $B_U$  common to all decision-makers that we are going to put in the same class. To this end, we may assume, for example, that  $B_C = \mathbb{B}_C$  and  $B_U = \mathbb{B}_U$ , where  $\mathbb{B}_C$  and  $\mathbb{B}_U$  are all possible preference relations on the sets  $C$  and  $U$  respectively.

As to the conditions imposed on the criterion choice rule, or on the projector  $\pi$ , it is not clear now whether such conditions may be found in our quite general formulation of the decision problem, i.e., in terms of preference relations. However, for some constraints on the class of decision situations, significant results have been obtained in this direction. These results constitute today's armory of decision theory.

The first significant constraint is that instead of the preference relations  $B$ , real-valued functions are used as the criterion for the ordering of the sets  $C$  and  $U$ . In economics these functions are called utility functions, and in engineering they are called loss functions. This restriction was sufficient to suggest an axiomatic description of four classes of projectors that brought forth four crite-

ria (by Wald, Savage, Hurwicz, and Laplace [20, 58]) uniquely for the matrix scheme  $Z_m$ , i.e., for the matrix situation model  $S_m$  with strict uncertainty.

However, the central result, the foundation of modern decision theory and its applications, particularly in economics, is the expected utility theorem [69, 61]. This theorem has played a principal role in microeconomics since the 1940s. Therefore, we consider it here with some comments but without proof (see the proof, for example, in [21, 61, 82]).

Suppose that in a decision system there is a nonparametric stochastic situation  $S_l = (Z_l, I_l)$ , where  $Z_l = (C, U, \{C_u \mid \forall u \in U\})$  and the regularity of the cause–effect mechanism  $I_l$  is the family of stochastic distributions  $\{Q_u, \forall u \in U\}$  on the set  $C$ . Let the decision-maker choose some complete, reflexive, and transitive binary preference relation  $\beta_C = (C, \succeq) \in B_C$  that orders the set of consequences  $C$ .

**Definition 2.5.** A real-valued function  $\mathfrak{U}$  defined on the ordered set  $C$  is a *utility function* if it is monotonic, i.e., if for all pairs  $(c_i, c_j)$ ,

$$c_i \succeq c_j \iff \mathfrak{U}(c_i) \geq \mathfrak{U}(c_j). \quad (2.21)$$

In order not to complicate this discussion we limit the presentation of this result to decision situation models with finite sets of decisions  $U$  and consequences  $C$ . Then, for example, the indices  $i$  of the ordered consequences  $c_1 \succeq c_2 \succeq \dots \succeq c_t$ ,  $\mathfrak{U}(c_i) = i, i = 1, 2, \dots, t$ , can serve as a utility function. Thus instead of the pair  $(S, \beta_C)$  we have the pair  $(S, \mathfrak{U}(\cdot))$ . But we need to construct a criterion of choice of optimal decision, that is, a utility function of decisions  $\hat{\mathfrak{U}}(\cdot)$ . In order to do this, first note that the set of probability distributions  $Q = \{Q_u, u \in U\}$  is homeomorphic to the set of decisions  $U$ , i.e.,  $Q_{u_i} \succ Q_{u_j}$  if and only if  $u_i \succ u_j$ .<sup>9</sup> Therefore, the set of probability distributions  $Q$ , or the set of lotteries, can substitute the set  $U$ . Suppose that the decision-maker would like her utility function on decisions to be linear on the set  $Q$  (i.e., on the set  $U$ ). To satisfy this demand, the preference relation on  $Q$  should satisfy the following conditions:

**Condition 3** For any  $Q_1, Q_2, Q_3 \in Q$  the sets

$$\{\alpha : \alpha Q_1 + (1 - \alpha)Q_2 \succeq Q_3\} \quad \text{and} \quad \{\alpha : Q_3 \succeq \alpha Q_1 + (1 - \alpha)Q_2\}$$

for  $\alpha \in [0, 1]$  are closed.

**Condition 4** For any  $Q_1, Q_2, Q_3 \in Q$ ,  $Q_1 \sim Q_2$ , and for any  $\alpha, 0 \leq \alpha \leq 1$ ,

$$\alpha Q_1 + (1 - \alpha)Q_3 \simeq \alpha Q_2 + (1 - \alpha)Q_3.$$

<sup>9</sup> In economics, the set  $Q$  is called the set of lotteries.

The first condition is the condition of continuity of the preference relation on  $\mathcal{Q}$ . The second is the condition of independence of the preference relation on  $\mathcal{Q}$ . Both of these conditions, conveying the wish of the decision-maker to have a linear decision utility function, are nevertheless purely mathematical, and could be common to many decision-makers. But what conditions can be characterized as specific to a certain group of decision-makers? Intuitively, it is clear that the preference relation on the set of lotteries  $\mathcal{Q}$  should be consistent with the preference relation on the set of consequences  $\mathcal{C}$ . One such consistency condition can be written, roughly speaking, in the following form:

**Condition 5** *If  $c_1 \succeq c_2$ , then*

$$\mathcal{Q} = \left( \left( \binom{c_1}{q_1} \right), \left( \binom{c_2}{q_2} \right) \right) \succeq \mathcal{Q}' = \left( \left( \binom{c_1}{q'_1} \right), \left( \binom{c_2}{q'_2} \right) \right) \quad (2.22)$$

*if  $q_1 > q'_1$ .*

That is, the decision-maker prefers the lottery in which the best consequence has the highest probability.

We now have the following theorem.

**Theorem 2.4.** *In order that a linear utility function  $\hat{\mathcal{U}}(u)$  exist in the form*

$$\hat{\mathcal{U}}(u) = \sum_{c \in \mathcal{C}} \mathcal{U}(c) q_u(c), \quad (2.23)$$

*it is necessary and sufficient that on the set  $U$ , Conditions 3–5 be satisfied. The utility function (2.23) is unique to within an increasing linear transformation.*

The utility (2.23) is called the *expected utility*. One can reformulate Theorem 2.4 in terms of a matrix situation. The expected utility function in this case is

$$\hat{\mathcal{U}}(u) = \sum_{\theta \in \Theta} \mathcal{U}(g(\theta, u)) p(\theta), \quad (2.24)$$

where  $p(\theta)$  and  $g(\theta, u)$  are found according to formulas (2.12) and (2.5) respectively.

In other words, in our framework, Conditions (or axioms) 3–5 describe a specific criterion choice rule or the class  $\Pi_0$  of projectors (or decision-makers)  $\pi$ . All decision-makers from this class would in the identical situation make decisions assuming that

$$\sum_{c \in \mathcal{C}} \mathcal{U}(c) q_{u_1}(c) \geq \sum_{c \in \mathcal{C}} \mathcal{U}(c) q_{u_2}(c) \Leftrightarrow u_1 \succeq u_2. \quad (2.25)$$

Note also that all conditions (axioms) of Theorem 2.4 presuppose that the regularity of the cause–effect mechanism of the decision situation is a stochastic

regularity in the form of a family of probability distributions  $I = \{Q_u, u \in U\}$ . Thus the preference relation (2.25) makes sense only for multiple, or mass, decisions under stochastic regularity.

Recall, though, that (2.25) takes place only over a very large number (infinite) of decisions in the same situation. When a single decision is made, the consequence is random with the probability distribution that corresponds to the chosen decision. This means that the consequence of a single choice of decision  $u_1$  can be worse than the consequence of a single choice of decision  $u_2$ . The same is true for all criteria belonging to the so-called nonexpected utility criteria group (see [59] and the literature therein), where the probability distribution on consequences is used in one way or another.

It seems that the only decision-maker who can avoid this is the decision-maker who makes decisions according to the criterion of Wald that sometimes is called *the principle of guaranteed result*.<sup>10</sup> In this case, the best decision has the form

$$u^0 = \arg \max_u \min_{\theta} L. \quad (2.26)$$

In other words, the decision-maker does not use any information about the regularity of the cause–effect mechanism. However, when in this situation decisions are made multiply (or massively), i.e., by many decision-makers but only once, or by one decision-maker but many times, this behavior, or attitude, seems less reasonable.<sup>11</sup>

The mass character of decisions in the axiomatics of the expected utility is presupposed by the regularity of the cause–effect mechanism in the form of a probability distribution. Namely, the third axiom (Condition 5) reflects this psychological element, demanding a decision-maker to prefer that decision where the most “useful” consequence has the greatest probability. In [68, p. 4] the following is said on this occasion: “Once you have introduced probabilities into the definition and measurement of utilities, you have made a bargain with the devil, and you can’t get rid of them again.”

We can ask whether we might be able to conjure up such an axiomatics of a criterion choice rule whereby the mass character of decisions does not rely on the regularity of the cause–effect mechanism. Perhaps in this way we can obtain a criterion that is good for decision-making in situations with regularities of mass phenomena different from stochastic ones. But are there such regularities, and what do we know about them?

To answer this question let us try—at first intuitively—to understand what phenomena it is natural to consider as random. The foremost difference between random and nonrandom phenomena consists in the fact that when we call some

<sup>10</sup> The axiomatics of this principle can be found, for example, in [20, 82].

<sup>11</sup> The author knew a professor who had the privilege of flying free of charge but never did so, fearing an accident.

phenomenon random, it always means that we do not know the regularities (call them local) that would allow us to predict the behavior of the phenomenon. The study of a random phenomenon can be twofold: one can reduce it to the nonrandom, looking for its local regularities, and one can, if it is a mass random phenomenon, try to find its global, statistical regularities, that is, regularities of the asymptotic behavior of the average values of different characteristics of the phenomenon. For example, this could be frequencies of certain consequences, arithmetic averages of certain functionals, etc.

If with the increase of the number of decisions and, correspondingly, of the number of appearances of their consequences, all these averages tend to certain limits (and some other similar conditions are satisfied as well; see details in [47]), then this phenomenon is called *statistically stable* or *stochastic*. And as is well known, the study of such phenomena is the subject matter of probability theory. At the same time, we can call all such phenomena where the aforementioned sample averages do not tend to unique limits *statistically unstable* random phenomena. It is reasonable to call these random phenomena *nonstochastic*. This takes place, for example, for the increasing average lifetime of a human being: for a newborn child, the probability of reaching the age of 60 has a tendency to increase, due to successes of medicine and hygiene.<sup>12</sup> Today such examples may be found in excess in economics and finance: time series of equity price indexes, interest rates, commodities, foreign exchange rates, etc.

It is natural to consider as *random in a broad sense* all mass phenomena that are studied only to within their statistical regularities. Once again, if a decision-maker happens to be in a situation with nonstochastic randomness, she cannot make use of the optimality criterion from Theorem 2.4. There are two reasons for this. First, the axioms of this theorem, i.e., the definition of the class  $\Pi_0$  of projectors, substantially use the regularity of stochastic randomness. Second, she does not know whether there exists any regularity of a nonstochastic random phenomenon.

Therefore two new problems appear. First, it is necessary to establish the existence of regularity of phenomena that are random in a broad sense. Second, it is necessary to find an axiomatic description of the class  $\Pi_1$  of decision-makers that would take into account only the mass character of a random phenomenon but not its regularity.

These two problems make up the core of the remainder of this book.

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<sup>12</sup> This example is due to Emile Borel [4].



# Chapter 3

## Indifferent Uncertainty

There are many faces to the mask of uncertainty.

---

Richard Bellman

### 3.1 Preliminaries

In the control of technological processes of mass production, one often finds the following decision situation. In every realization of a technological operation, it is necessary to choose the value of some operational parameter, for example, the time  $t$  of a treatment, knowing the following information:

- (1) The “ideal” value  $t^*$  of the parameter  $t$ —that is, the value that we would choose if we knew all necessary variables or inputs—is contained within certain limits  $t_1 \leq t^* \leq t_2$ .
- (2) If  $t$  deviates from  $t^*$  by no more than a given value  $\Delta t > 0$ , the article is good; otherwise it is bad or wasted.
- (3) The technological operation is carried out many times; therefore it is expedient to estimate it by the average quantity of wasted articles over infinitely many realizations.

It is clear that if

$$\Delta t < \frac{t_2 - t_1}{2},$$

then there is no hope of obtaining all products in good condition, and now we have a problem of how to decrease the percentage of waste. This problem is often formulated in the following way.

To estimate one operation, choose the loss function in the form

$$l(t^*, t) = \begin{cases} 0, & \text{if } |t^* - t| \leq \Delta t, \\ 1, & \text{if } |t^* - t| > \Delta t, \end{cases} \quad \text{with } t^*, t \in [t_1, t_2].$$

Denoting by  $\bar{t}^* = (t_1^*, t_2^*, \dots)$  the sequence of unknown “ideal” values of the parameter, and by  $\bar{t} = (t_1, t_2, \dots)$  the sequence of values that we choose (our decisions or controls), we introduce the function

$$L(\bar{t}^*, \bar{t}) = \limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N l(t_n^*, t_n).$$

This function serves as an estimate of the control sequence  $\bar{t}$  for infinitely many technological operations. We shall further call it the average loss function. In the situation under consideration our wish to reach the minimal value of the average loss function—our goal—by means of the choice of “control”  $\bar{t}$  appears to be quite natural. However, dependence of the average loss function on the *unknown* sequence  $\bar{t}^*$  prevents us from achieving this goal. Speaking figuratively, for any chosen  $\bar{t}$  the value of the average loss function is *uncertain*.

At first sight, it seems that it is possible to eliminate this uncertainty by means of reduction of the control problem to the following optimization problem:

$$\sup_{\bar{t}^*} L(\bar{t}^*, \bar{t}) \rightarrow \inf_{\bar{t}} \left( \bar{t}^*, \bar{t} \in [t_1, t_2]^{\mathbb{N}} \right), \quad (3.1)$$

where  $[t_1, t_2]^{\mathbb{N}}$  is the set of all possible sequences with values in  $[t_1, t_2]$ . But two objections appear. First, in the case

$$\Delta t < \frac{t_2 - t_1}{2},$$

we arrive at an absurd conclusion: the left-hand side of (3.1) will be completely independent of  $\bar{t}$ , being identical to 1. As a result, we are deprived of the possibility of preferring one control to another. Second, such a reduction of the problem corresponds to what would be the case if  $\bar{t}^*$  were chosen by a reasonable, intelligent adversary completely informed about our choice of  $\bar{t}$ . If one could link the first objection with the form of the loss function especially selected for our situation (which, as a matter of fact, is not the case), the second objection would have a deeper and more general character. The fact is that in control of technological processes the assumption that the choice of  $\bar{t}^*$  is made by an intelligent adversary is unrealistic: there is no such opponent. Having admitted his existence, assuming that

$$\bar{t}^* \in [t_1, t_2]^{\mathbb{N}},$$

we have deliberately overestimated the minimal percentage of waste, accepting at the same time some unknown and groundless additional losses. Of course,

in technological control problems this assumption is unrealistic. Yet still worse is that such a direct approach not only leads to an overestimation of losses, but also, and this is of principal importance in the considered problem, provides us with no recommendation with respect to the choice of  $\bar{t}$ . This issue, appearing in the “game against nature,” is well known in decision theory. There are different recommendations as to how to overcome it. According to [30], one has to choose some  $0 < \alpha < 1$  and to minimize not

$$\sup_{\bar{t}^* \in [t_1, t_2]^{\mathbb{N}}} L(\bar{t}^*, \bar{t}),$$

but the value

$$\alpha \inf_{\bar{t}^* \in [t_1, t_2]^{\mathbb{N}}} L(\bar{t}^*, \bar{t}) + (1 - \alpha) \sup_{\bar{t}^* \in [t_1, t_2]^{\mathbb{N}}} L(\bar{t}^*, \bar{t}).$$

Unfortunately, in this approach the choice of  $\alpha$  remains a riddle. The aim of this and other empirical criteria that are more or less commonly used in decision theory (see, for example, [58, Chapter 13], [59]) is not a reflection of objectively existing information about the unknown parameter, but more likely a clarification of the intuition of the decision-maker. Therefore, for control of technological processes such “psychological” means are unconvincing.

What remains? Either

- (1) the assumption that  $\bar{t}_1^*, \bar{t}_2^*, \dots$  is the sequence of values of independent random variables with an unknown probabilistic distributions  $P$ , or
- (2) the guaranteed result principle, formulated in its most general form in [22]. In case of the considered problem, this principle consists in extracting the set  $\Gamma$  of “situations”  $(\bar{t}^*, \bar{t})$  that we consider “real” or “permissible” and to minimize instead of the value

$$\sup_{\bar{t}^* \in [t_1, t_2]^{\mathbb{N}}} L(\bar{t}^*, \bar{t}),$$

the value

$$\sup_{\bar{t}^* \in \Gamma(\bar{t})} L(\bar{t}^*, \bar{t}), \quad \text{where } \Gamma(\bar{t}) = \left\{ \bar{t}^* \in [t_1, t_2]^{\mathbb{N}} : (\bar{t}^*, \bar{t}) \in \Gamma \right\}.$$

The first of these two approaches is mathematically the clearest and allows us to take advantage of all the modern arsenal of antagonistic game theory (see, for example, [84]). However, in many practical situations the assumption of existence of a probabilistic distribution is more than doubtful. Indeed, this assumption imposes statistical uniformity of behavior of the sequence  $\bar{t}^*$ . It is clear that in many cases one cannot guarantee this: frequencies of various values can differ considerably from sample to sample. Of course, one can object that from a practical

point of view, all mathematical details of asymptotic behavior of frequencies are unimportant, and what really matters is only what happens in finite samples. But this objection is incorrect. In passing to consideration of mixed strategies, game theory essentially uses the probability-theoretic notion of independence, which is not transferred to the case of finite samples.

The theory of noncoalition (nonantagonistic) games [84], currently under active development, also cannot be applied directly to the problem under consideration. We meet the same difficulties as with the “psychological,” or empirical, criteria mentioned above: we can nowhere obtain information for the precise assignment of the goal that we formally ought to attribute here to the physical mechanism generating the sequence  $\vec{t}^*$ .

Therefore, from all currently existing approaches, only the guaranteed result principle has a chance to be adequately applied in the situation. But first, we must understand which sequences  $\vec{t}^*$  we are ready to accept as real, and “not especially invented as a response on our  $\vec{t}$ ,” and second, we have to define mathematically the set of all such sequences. It turns out that this can be done with very little additional information. Indeed, if the physical mechanism generating the sequence  $\vec{t}^*$  is one of the manifestations of the indifference (with respect to our purposes) of nature, then it is natural to suppose that each member of the sequence  $\vec{t}^*$  appears without any relation to the members of the sequence  $\vec{t}$ , independently and irrespectively of them. In problems of control of technological processes, the possibility of such parameterization follows from the essence of the production organization. It seems that this can be true in many other situations.

Let us introduce more general notation. Let  $X$  and  $Y$  be arbitrary nonempty sets, and  $\Theta = X^{\mathbb{N}}$ ,  $U = Y^{\mathbb{N}}$  the sets of all possible sequences of their elements. What conditions must the sequences  $\theta \in \Theta$  and  $u \in U$  (they play further the roles of the respective sequences  $\vec{t}^*$  and  $\vec{t}$  described above) satisfy in order that we could believe that each of them was built without regard to the other? If the sets  $X$  and  $Y$  are finite, probability theory prompts a natural variant of the required condition: the conditional frequencies of appearance of different elements  $x \in X$  in the sequence  $\theta$  must not depend on the “conditions,” i.e., on what elements  $y \in Y$  simultaneously appear in the sequence  $u$ . Mathematically it is more convenient here, as in probability theory, to write down this condition in the form

$$\lim_{N \rightarrow \infty} \left[ \frac{1}{N} \sum_{n=1}^N \mathbf{1}_x(\theta_n) \mathbf{1}_y(u_n) - \left( \frac{1}{N} \sum_{n_1=1}^N \mathbf{1}_x(\theta_{n_1}) \right) \left( \frac{1}{N} \sum_{n_2=1}^N \mathbf{1}_y(u_{n_2}) \right) \right] = 0, \quad (3.2)$$

$$\forall x \in X, y \in Y.$$

Here and in what follows, the symbol  $\mathbf{1}_A(\cdot)$  denotes the characteristic function of the set  $A$  (perhaps a single point).

Declaring independent those and only those pairs of sequences  $(\theta, u)$  that satisfy relation (3.2), we can intuitively formulate the following statements:<sup>1</sup>

- (1) If one of the sequences  $\theta$  and  $u$  is stationary, for example  $\theta_1 = \theta_2 = \dots$ , then regardless of the other sequence  $u$ , the sequences  $\theta$  and  $u$  are independent in the sense of (3.2).
- (2) If one or both sequences  $\theta$  and  $u$  are realizations of sequences of independent random variables, then with probability 1 these sequences are independent in the sense of (3.2).
- (3) If one of the two sequences  $\theta$  and  $u$ , for example  $\theta$ , is a realization of a sequence of independent random values and the other is an arbitrary fixed sequence, then with probability 1 these sequences are independent in the sense of (3.2).

These statements confirm that the definition we have introduced is quite natural; yet there is an objection. Let the sequence  $u$ , interpreted as a sequence of control parameter values or as a sequence of decisions, be fixed. How can we write down the condition that the behavior of the uncontrolled parameter  $\theta$  does not depend on  $u$ ? Obviously, in the form

$$\theta \in \{\theta' \in \Theta : \text{the relation (3.2) is satisfied}\}.$$

But formally, this looks like dependence! To see the weakness of this objection we can imagine the following situation. Two blindfolded persons are each asked to choose a point, that is, to choose  $\theta$  and  $u$ , in the segment  $[0, 1]$ . Can we believe that under these conditions they could possibly choose the same point? Of course not. But in formal language, the situation looks analogous to our definition of independence

$$\theta \in \{\theta' \in [0, 1] : \theta' \neq u\}.$$

So formal dependence of the permissible set of values of one variable on the values of another variable is not yet what we intuitively understand as dependence. It is the “size” of the forbidden area that matters, and in our case one can conclude on the basis of the above statements (1)–(3) that this area is sufficiently small.

If the sets  $X$  and  $Y$  are arbitrary, selection of the set of independent pairs of sequences is more complicated. One can immediately see that exact transfer of the relation (3.2) in this case is impossible: any two sequences (even sequences related by functional dependence) that take each value no more than once would be independent. At the same time, to proceed as in probability theory, that is, to select some systems of subsets  $\mathcal{A}$  and  $\mathcal{B}$  in  $X$  and  $Y$  respectively and to formulate a definition like (3.2) but this time with respect to the sets from these systems, is not an easy task.

<sup>1</sup> Their validity follows from general theorems that are proved below, in Section 3.2.

The point is that the choice of such systems of subsets cannot be completely arbitrary. For example, taking as  $\mathcal{A}$  and  $\mathcal{B}$  the one-element systems consisting of  $X$  and  $Y$  themselves, i.e.,

$$\mathcal{A} = \{X\}, \quad \mathcal{B} = \{Y\},$$

we find that any two sequences are independent. On the other hand, one can show that even if  $\theta$  and  $u$  are sequences of values of independent random variables having continuous distribution functions, with probability 1 there exist systems of subsets  $\mathcal{A}$  and  $\mathcal{B}$  such that with respect to them the sequences  $\theta$  and  $u$  are dependent.

It seems that the natural way out of this situation consists in constructing the required systems of subsets using directly the loss function  $l$ . In this case, instead of “independence” we shall speak about “ $\Gamma_l$ -independence,” which in some sense is even more convenient: for example, in the case of finite  $X$  and  $Y$ , it can be that dependent sequences become  $\Gamma_l$ -independent for some loss function  $l$  [36].

Therefore we introduce the following definition of  $\Gamma_l$ -independence.

Let  $X$  and  $Y$  be arbitrary subsets,  $l : X \times Y \rightarrow \mathbb{R}$  a bounded real function, and  $\mathbb{N}$  the set of natural numbers. Furthermore, we shall denote the union of mutually disjoint sets by the plus sign.

**Definition 3.1.** We shall say that the sequences  $\theta \in (\mathbb{N} \rightarrow X) = \Theta$  and  $u \in (\mathbb{N} \rightarrow Y) = U$  are  $\Gamma_l$ -independent, or that they are in relation  $\Gamma_l$  ( $\Gamma_l \subset U \times \Theta$ ), if for any  $\varepsilon > 0$  one can determine finite subdivisions

$$X = \sum_{i \in I} X_i, \quad Y = \sum_{j \in J} Y_j, \quad I \times J = E_0 + E_1,$$

of sets  $X$ ,  $Y$ , and  $I \times J$  into disjoint subsets such that the following conditions are satisfied:

**Condition 1**

$$\sup_{\substack{x_1, x_2 \in X_i \\ y_1, y_2 \in Y_j}} |l(x_1, y_1) - l(x_2, y_2)| < \varepsilon \quad \forall (i, j) \in E_1;$$

**Condition 2** For all  $i \in I$ ,  $j \in J$ ,

$$\lim_{N \rightarrow \infty} \left[ \frac{1}{N} \sum_{n=1}^N l_{X_i}(\theta_n) l_{Y_j}(u_n) - \frac{1}{N^2} \sum_{n_1=1}^N l_{X_i}(\theta_{n_1}) \sum_{n_2=1}^N l_{Y_j}(u_{n_2}) \right] = 0;$$

**Condition 3**

$$\lim_{N \rightarrow \infty} \sup_{(i,j) \in E_0} \sum_{n=1}^N \frac{1}{N} l_{X_i}(\theta_n) l_{Y_j}(u_n) < \varepsilon.$$

We note that  $(\theta, u) \in \Gamma_l$  will also be written in the form  $\theta \in \Gamma_l(u)$ . Theorems 3.1 and 3.2 confirm that this definition is natural. Now our problem can be written as

$$\sup_{\theta \in \Gamma_l(u)} L(\theta, u) \rightarrow \inf_{u \in U}. \quad (3.3)$$

Later on, we shall call the problem (3.3) a *decision-making problem under  $\Gamma_l$ -independence* (it is implied that the sets  $X, Y$  are included in the definition of the function  $l$ ).

Solution of the problem (3.3) is based on the following principal identity (Theorem 3.3):

$$\sup_{\theta \in \Gamma_l(u)} L(\theta, u) = \sup_{x \in X} L(\theta_x, u) \quad \forall u \in U,$$

where  $\theta_x$  is a stationary sequence  $\theta_1 = \theta_2 = \dots = x$ . This can be interpreted in such a way that the changes of  $\theta$  that take place “regardless of  $u$ ” are not “dangerous” to us.

It is proved further that under broad assumptions,

$$\inf_{u \in U} \sup_{x \in X} L(\theta_x, u) = \inf_{q \in \mathcal{P}_{\sigma}(Y)} \sup_{p \in \mathcal{P}_{\sigma}(X)} \int l(x, y) p(dx) q(dy),$$

where  $\mathcal{P}_{\sigma}(\cdot)$  is the family of all probability distributions on the corresponding  $\sigma$ -algebra of subsets of the set  $(\cdot)$ .

As a result, the control problem for  $\Gamma_l$ -independence is reduced to some antagonistic game, and we come to Theorems 3.4, 3.5, 3.7. Roughly speaking, the sense of these theorems is that the minimal average losses in the case of  $\Gamma_l$ -independence turn out to be equal to the value of this game. At the same time, we note an interesting difference from the game theory: it turns out that the optimal sequence of decisions  $u$  does not have to be stochastic even in the case that the optimal strategy in the appropriate game is mixed. It is sufficient that frequencies of different values of the chosen sequence tend, in a certain sense, to the optimal mixed strategy. This is fairly consistent with the initial assumption that distinguishes the considered decision or control problem from the game problem: there is no antagonist here who is capable of using the knowledge of our algorithm against us.

### 3.2 $\Gamma_l$ -independent Sequences

Unless otherwise stated, we everywhere suppose that the following assumptions hold:

1.  $X$  and  $Y$  are compact Hausdorff spaces with respective countable bases  $\mathbf{A}$  and  $\mathbf{B}$ ;
2.  $\mathcal{A}$  and  $\mathcal{B}$  are respectively  $\sigma$ -algebras of Borel sets of the spaces  $X$  and  $Y$ ;
3.  $\mathcal{P}(X)$  and  $\mathcal{P}(Y)$  are respectively families of all probability distributions on  $\mathcal{A}$  and  $\mathcal{B}$ ;
4.  $\xi_n$  and  $\eta_n$ ,  $n \in \mathbb{N}$ , are independent random elements taking values in  $X$  and  $Y$  respectively; the distributions of all  $\xi_n$  are equal and denoted by  $\mu$ , while the distributions of all  $\eta_n$  are equal as well, and denoted by  $\nu$ ;
5. the joint distribution of the pair of independent random elements  $(\xi_n, \eta_n)$  is denoted by  $\lambda$  ( $\lambda = \mu \cdot \nu$ );
6. finally,  $C$  is the set of all break points (which may be empty) of the function  $l$ , and  $\mathbf{G}$  is a family of all open sets  $G \subseteq X \times Y$  covering  $C$ .

**Theorem 3.1.** *If  $\inf_{G \in \mathbf{G}} \lambda(G) = 0$ , then  $((\eta_n)_{n \in \mathbb{N}}, (\xi_n)_{n \in \mathbb{N}}) \in \Gamma_l$  with probability 1.*

To prove this and other theorems we need the following lemma.

**Lemma 3.1.** *Let  $S$  be a finite set of indexes,  $X$  and  $Y$  arbitrary sets, and  $X^{(s)} \subseteq X$ ,  $Y^{(s)} \subseteq Y$ ,  $s \in S$ . Then one can find finite subdivisions of the sets*

$$X = \sum_{i \in I} X_i, \quad Y = \sum_{j \in J} Y_j, \quad I \times J = E_{(0)} + E_{(1)}$$

*into disjoint subsets such that the following conditions are satisfied:*

**Condition 1**

$$\bigcup_{s \in S} X^{(s)} \times Y^{(s)} = \sum_{(i,j) \in E_1} X_i \times Y_j;$$

**Condition 2**

$$(X \times Y) \setminus \left( \bigcup_{s \in S} X^{(s)} \times Y^{(s)} \right) = \sum_{(i,j) \in E_0} (X_i \times Y_j);$$

**Condition 3** *For any  $s \in S$  one can find subsets of indices  $I_s \in I$  and  $J_s \in J$  such that*

$$X^{(s)} = \sum_{i \in I_s} X_i, \quad Y^{(s)} = \sum_{j \in J_s} Y_j.$$

**Condition 4** *For arbitrary  $i \in I$ ,  $j \in J$ , the sets  $X_i$  and  $Y_j$  belong to algebras of sets generated by the respective families*

$$\left\{ X^{(s)} \mid s \in S \right\} \quad \text{and} \quad \left\{ Y^{(s)} \mid s \in S \right\}.$$

*Proof.* Let

$$\alpha = \left\{ (x_1, x_2) \in X \times X : \forall s \in S \quad (x_1 \in X^{(s)} \iff x_2 \in X^{(s)}) \right\},$$

$$\beta = \left\{ (y_1, y_2) \in Y \times Y : \forall s \in S \quad (y_1 \in Y^{(s)} \iff y_2 \in Y^{(s)}) \right\}.$$

Obviously, the relations  $\alpha$  and  $\beta$  are relations of equivalence that divide the sets  $X$  and  $Y$  into disjoint of equivalence classes

$$X = \sum_{i \in I} X_i, \quad Y = \sum_{j \in J} Y_j,$$

and since the set  $S$  is finite, the number of such classes is finite. Define

$$Z = \bigcup_{s \in S} X^{(s)} \times Y^{(s)},$$

$$E_0 = \{(i, j) \in I \times J : X_i \times Y_j \cap Z = \emptyset\},$$

$$E_1 = (I \times J) \setminus E_0,$$

and prove that in this case, Conditions 1–4 are satisfied.

Note that for any  $i \in I$ , either it is possible to find a nonempty subset  $S_i \subseteq S$  such that

$$X_i = \bigcap_{s \in S_i} X^{(s)}$$

or else

$$X_i = X \setminus \bigcup_{s \in S} X^{(s)},$$

and analogously for any  $j \in J$ . This proves Condition 4.

Further, if  $(x, y) \in Z$ , then one can find a pair  $(i, j) \in I \times J$  such that  $(x, y) \in X_i \times Y_j$ , and since  $(x, y) \in Z$ , it is clear that  $(i, j) \in E_1$ . In other words,

$$Z \subseteq \sum_{(i, j) \in E_1} X_i \times Y_j.$$

Conversely, if

$$(i, j) \in E_1, \quad (x, y) \in X_i \times Y_j,$$

then from

$$(X_i \times Y_j) \cap Z \neq \emptyset$$

it follows that the point

$$(x_1, y_1) \in (X_i \times Y_j) \cap Z$$

exists as well, which means that there exists a nonempty set  $S_1 \subseteq S$  for which

$$(x_1, y_1) \in \bigcap_{s \in S_1} X^{(s)} \times Y^{(s)}.$$

In this case,

$$x_1 \in \bigcap_{s \in S_1} X^{(s)}, \quad y_1 \in \bigcap_{s \in S_1} Y^{(s)},$$

and thus

$$x \in \bigcap_{s \in S_1} X^{(s)}, \quad y \in \bigcap_{s \in S_1} Y^{(s)},$$

whence it follows that

$$(x, y) \in \bigcap_{s \in S_1} X^{(s)} \times \bigcap_{s \in S_1} Y^{(s)} \subseteq \bigcap_{s \in S_1} (X^{(s)} \times Y^{(s)}) \subseteq Z.$$

Condition 1 has been proved. The validity of Condition 2 follows from the validity of Condition 1 and from the equality

$$\sum_{(i,j) \in E_0} X_i \times Y_j = (X \times Y) \setminus \sum_{(i,j) \in E_1} (X_i \times Y_j).$$

Finally, Condition 3 means just that for any  $i \in I, s \in S$ ,

- either  $X_i \subseteq X^{(s)}$
- or  $X_i \cap X^{(s)} = \emptyset$ ,

and analogously,

- either  $Y_j \subseteq Y^{(s)}$
- or  $Y_j \cap Y^{(s)} = \emptyset$  for any  $j \in J$ .

And this is indeed so, for otherwise, if, for example,

$$X_i \cap X^{(s)} \neq \emptyset \quad \text{and} \quad X_i \setminus X^{(s)} \neq \emptyset,$$

then points  $x_1, x_2 \in X_i$  could be found such that  $x_1 \in X^{(s)}$  and  $x_2 \notin X^{(s)}$ , in spite of the choice of  $X_i$  as the equivalence class modulo  $\alpha$ . The lemma is proved.

We now proceed to the proof of Theorem 3.1.

*Proof (of Theorem 3.1).* Choosing an arbitrary  $\varepsilon > 0$ , fix the set  $G_\varepsilon \in \mathbf{G}$  such that  $\lambda(G_\varepsilon) < \varepsilon$ . Then for its complement

$$F_\varepsilon = (X \times Y) \setminus G_\varepsilon,$$

the inequality

$$\lambda(F_\varepsilon) > 1 - \varepsilon$$

holds, and the function  $l$  will be continuous at all points of the set  $F_\varepsilon$ . Let us surround every point  $\mathbf{z} = (x, y)$  of the set  $F_\varepsilon$  by a neighborhood of the form

$$Q_z = A \times B \quad (x \in A \in \mathbf{A}, \quad y \in B \in \mathbf{B}),$$

so that the following relation is satisfied:

$$\text{diam}(l(Q_z)) \stackrel{\text{def}}{=} \sup_{z_1, z_2 \in Q_z} |l(z_1) - l(z_2)| < \varepsilon.$$

From the open cover  $\{Q_z | z \in F_\varepsilon\}$  of the closed and hence compact set  $F_\varepsilon$ , let us choose a finite subcover

$$\{Q^{(s)} | s \in S\},$$

and making use of Lemma 3.1, construct subdivisions

$$X = \sum_{i \in I} X_i, \quad Y = \sum_{j \in J} Y_j, \quad I \times J = E_0 + E_1$$

corresponding to the family

$$\{Q^{(s)} | s \in S\}.$$

In so doing, we discover that Condition 1 of  $\Gamma$ -independence is satisfied by construction of the subdivisions  $X_i \in \mathcal{A}$  and  $Y_j \in \mathcal{B}$  for all  $(i, j) \in I \times J$ , and therefore, according to the strong law of large numbers, Condition 2 is satisfied with probability 1 for the sequence of values  $(\xi_n)_{n \in \mathbb{N}}$  and  $(\eta_n)_{n \in \mathbb{N}}$ .

Finally,

$$\begin{aligned} \sum_{(i,j) \in E_0} \frac{1}{N} \sum_{n=1}^N \mathbf{1}_{X_i}(\xi_n) \mathbf{1}_{Y_j}(\eta_n) &= \frac{1}{N} \sum_{n=1}^N \mathbf{1}_{Z_0}(\xi_n, \eta_n) \\ &\leq \frac{1}{N} \sum_{n=1}^N \mathbf{1}_{Z \setminus F_\varepsilon}(\xi_n, \eta_n) \xrightarrow{a.c.} \lambda(Z \setminus F_\varepsilon) < \varepsilon, \end{aligned}$$

where

$$Z_0 = \sum_{(i,j) \in E_0} X_i \times Y_j, \quad Z = X \times Y,$$

so that Condition 3 is satisfied with probability 1. Theorem 3.1 is proved.

**Theorem 3.2.** *If  $u \in U$  and*

$$\inf_{G \in \mathbf{G}} \sup_{y \in Y} \mu \{x \in X : (x, y) \in G\} = 0,$$

*then  $(u, (\xi_n)_{n \in \mathbb{N}}) \in \Gamma_1$  with probability 1.*

*Proof.* Choosing an arbitrary  $\varepsilon > 0$ , we find a set  $G \in \mathbf{G}$  such that

$$\sup_{y \in Y} \mu \{x \in X : (x, y) \in G_\varepsilon\} < \varepsilon,$$

and then making use of this set, we construct the set  $F_\varepsilon$  and the subdivisions

$$X = \sum_{i \in I} X_i, \quad Y = \sum_{j \in J} Y_j, \quad I \times J = E_0 + E_1$$

in the same way as in the proof of Theorem 3.1. Let us show that in this case all the conditions of  $\Gamma_1$ -independence will be satisfied with probability 1.

Condition 1 is satisfied by construction. To prove that Conditions 2 and 3 are satisfied, we define

$$\begin{aligned} H_j(N) &= \{n \leq N : u_n \in Y_j\}, \\ k_j(N) &= \sum_{n=1}^N \mathbf{1}_{Y_j}(u_n) = \text{Card}(H_j(N)), \\ Z_{i,j} &= (X_i \times Y_j), \\ \zeta_{i,j}(N) &= \begin{cases} [\sum_{n=1}^N \mathbf{1}_{Z_{i,j}}(\xi_n, u_n)] / k_j(N), & \text{if } k_j(N) \neq 0; \\ \mu(X_i), & \text{if } k_j(N) = 0. \end{cases} \end{aligned}$$

It is obvious that

$$\sum_{n=1}^N \mathbf{1}_{Z_{i,j}}(\xi_n, u_n) = \sum_{n \in H_j(N)} \mathbf{1}_{Z_{i,j}}(\xi_n, u_n) = \sum_{n \in H_j(N)} \mathbf{1}_{X_i}(\xi_n),$$

and therefore, according to the strong law of large numbers,

$$\zeta_{ij}(N) \xrightarrow{a.c.} \mu(X_i) \tag{3.4}$$

for all  $i \in I, j \in J$ . Thus

$$\begin{aligned} & \frac{1}{N} \sum_{n=1}^N \mathbf{1}_{Z_{i,j}}(\xi_n, u_n) - \frac{1}{N^2} \sum_{n_1=1}^N \mathbf{1}_{X_i}(\xi_{n_1}) \sum_{n_2=1}^N \mathbf{1}_{Y_j}(u_{n_2}) \\ &= \frac{k_j(N)}{N} \left[ \zeta_{ij}(N) - \frac{1}{N} \sum_{n_1=1}^N \mathbf{1}_{X_i}(\xi_{n_1}) \right] \xrightarrow{a.c.} 0 \end{aligned}$$

for all  $i \in I$ ,  $j \in J$ . That is, Condition 2 is satisfied with probability 1.

It remains to prove that the same is true with respect to Condition 3. To this end, note first that if  $\tilde{\nu}$  is some probability on an arbitrary  $\sigma$ -algebra  $\mathcal{B}$  of subsets of the set  $Y$ , the set  $Z_0$  belongs to the  $\sigma$ -algebra  $\mathcal{A} \otimes \mathcal{B}$ , and  $\tilde{\lambda} = \mu \tilde{\nu}$ , then  $\tilde{\lambda}(Z_0) < \varepsilon$ . Indeed, by Fubini's theorem,

$$\tilde{\lambda}(Z_0) = \int_Y \mu(\tilde{X}_y) \tilde{\nu}(dy),$$

where

$$\tilde{X}_y = \{x \in X : (x, y) \in Z_0\},$$

and since  $Z_0 \subseteq G_\varepsilon$ , from

$$\mu\{x \in X : (x, y) \in G_\varepsilon\} < \varepsilon$$

we conclude that

$$\tilde{\lambda}(Z_0) < \varepsilon.$$

We now introduce into consideration the probability  $\tilde{\nu}_N$  on the  $\sigma$ -algebra  $\tilde{\mathcal{B}}$  generated by the family of sets  $\{Y_j | j \in J\}$ , which we define by the formula

$$\tilde{\nu}(Y_j) = \frac{1}{N} \sum_{n=1}^N \mathbf{1}_{Y_j}(u_n).$$

It follows from (3.4) that with probability 1,  $N_0 \in \mathbb{N}$  can be found such that for all  $N > N_0$ , the following inequality will be satisfied:

$$|\zeta_{ij}(N) - \mu(X_i)| < \frac{\varepsilon}{\text{Card}(I)}$$

for all  $(i, j) \in E_0$  and therefore for all  $N$  such that

$$\begin{aligned}
& \left| \frac{1}{N} \sum_{n=1}^N \mathbf{1}_{Z_0}(\xi_n, u_n) - \sum_{(i,j) \in E_0} \mu(X_i) \tilde{v}(Y_j) \right| \\
&= \left| \sum_{(i,j) \in E_0} \zeta_{i,j}(N) \tilde{v}(Y_j) - \sum_{(i,j) \in E_0} \mu(X_i) \tilde{v}_N(Y_j) \right| \\
&< \frac{\varepsilon}{\text{Card}(I)} \sum_{(i,j) \in E_0} \tilde{v}_N(Y_j) < \varepsilon
\end{aligned}$$

with probability 1. This, according to the above comment, means that the following inequality will hold with probability 1:

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \mathbf{1}_{Z_0}(\xi_n, u_n) + \sum_{(i,j) \in E_0} \mu(X_i) \tilde{v}(Y_j) = \tilde{\lambda}(Z_0) + \varepsilon < 2\varepsilon,$$

where  $\tilde{\lambda}_N = \mu \cdot \tilde{v}_N$ . By virtue of the arbitrariness of  $\varepsilon > 0$ , this completes the proof of Theorem 3.2.

### 3.3 Main Identity

Define

$$\begin{aligned}
L &: \Theta \times U \rightarrow R, \\
L(\theta, u) &= \limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N l(\theta_n, u_n), \\
\mathcal{K}_l(u) &= \sup_{\theta \in \Gamma_l(u)} L(\theta, u) \quad \forall u \in U.
\end{aligned}$$

#### Theorem 3.3.

$$\mathcal{K}_l(u) = \sup_{x \in X} \limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N l(x, u_n), \quad \forall u \in U.$$

*Proof ([155]).* Denote by  $\bar{x}$  the stationary sequence  $(x, x, \dots)$ , where  $x \in X$ , and by  $\bar{X}$  the set of all stationary sequences from  $\Theta$ . Obviously, whatever the sequence  $u \in U$ , in consequence of Theorem 3.2, assuming that  $\xi_n$  is concentrated in one point  $x$ , it is always true that  $\bar{x} \in \Gamma_l(u)$ , and therefore

$$\mathcal{K}_l(u) = \sup_{\theta \in \Gamma_l(u)} L(\theta, u) \geq \sup_{x \in X} L(\bar{x}, u) = \sup_{x \in X} \limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N l(x, u_n),$$

so it is sufficient to prove that for  $\theta \in \Gamma_n(u)$  the inequality

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N l(\theta_n, u_n) \leq \sup_{x \in X} \limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N l(x, u_n) \quad (3.5)$$

holds. Given an arbitrary  $\varepsilon > 0$ , we construct the subdivisions

$$X = \sum_{i \in I} X_i, \quad Y = \sum_{j \in J} Y_j, \quad I \times J = E_0 + E_1$$

that appear in the definition of  $\Gamma_l$ , and setting

$$X_i \times Y_j = Z_{ij}, \quad X \times Y = Z, \quad \sum_{(i,j) \in E_k} Z_{ij} = Z_k \quad (k = 0, 1),$$

we shall assume, without loss of generality, that

$$Z_{ij} \neq \emptyset, \quad \text{when } (i, j) \in E_1.$$

Then to every pair of indices  $(i, j) \in I \times J$  we associate some point

$$z_{ij} = (x_i, y_j) \in X \times Y$$

such that  $z_{ij} \in Z_{ij}$  when  $(i, j) \in E_1$ . Let  $(u, \theta) \in \Gamma_l$ . Now represent the expression

$$\bar{l}(N) = \frac{1}{N} \sum_{n=1}^N \bar{l}(\theta_n, u_n)$$

in the form of the sum

$$\bar{l}(N) = \sum_{k=1}^6 \bar{l}_k(N),$$

where

$$\bar{l}_1(N) = \sum_{(i,j) \in E_0} \frac{1}{N} \sum_{n=1}^N \mathbf{1}_{Z_{ij}}(\theta_n, u_n) [l(\theta_n, u_n) - l(x_i, y_j)];$$

$$\bar{l}_2(N) = \sum_{(i,j) \in E_1} \frac{1}{N} \sum_{n=1}^N \mathbf{1}_{Z_{ij}}(\theta_n, u_n) [l(\theta_n, u_n) - l(x_i, y_j)];$$

$$\bar{l}_3(N) = \sum_{\substack{i \in I, \\ j \in J}} l(x_i, y_j) \left[ \frac{1}{N} \sum_{n=1}^N \mathbf{1}_{Z_{ij}}(\theta_n, u_n) - \frac{1}{N^2} \sum_{n_1=1}^N \mathbf{1}_{X_i}(\theta_{n_1}) \sum_{n_2=1}^N \mathbf{1}_{Y_j}(u_{n_2}) \right];$$

$$\bar{l}_4(N) = \sum_{(i,j) \in E_0} \frac{1}{N^2} \sum_{n_1=1}^N \mathbf{1}_{X_i}(\theta_{n_1}) \sum_{n_2=1}^N \mathbf{1}_{Y_j}(u_{n_2}) [l(x_i, y_j) - l(x_i, u_{n_2})];$$

$$\bar{l}_5(N) = \sum_{(i,j) \in E_1} \frac{1}{N^2} \sum_{n_1=1}^N \mathbf{1}_{X_i}(\theta_{n_1}) \sum_{n_2=1}^N \mathbf{1}_{Y_j}(u_{n_2}) [l(x_i, y_j) - l(x_i, u_{n_2})];$$

$$\bar{l}_6(N) = \sum_{\substack{i \in I \\ j \in J}} \frac{1}{N^2} \sum_{n_1=1}^N \mathbf{1}_{X_i}(\theta_{n_1}) \sum_{n_2=1}^N \mathbf{1}_{Y_j}(u_{n_2}) l(x_i, u_{n_2}).$$

The following relations can be checked easily:

$$|\bar{l}_1(N)| \leq \text{diam}(l(Z_0)) \frac{1}{N} \sum_{n=1}^N \mathbf{1}_{Z_0}(\theta_n, u_n) < (\varepsilon + o(1)) \text{diam}(l(Z_0))$$

for  $N \rightarrow \infty$ ;

$$|\bar{l}_2(N)| < \varepsilon \sum_{(i,j) \in E_1} \frac{1}{N} \sum_{n=1}^N \mathbf{1}_{Z_{ij}}(\theta_n, u_n) \leq \varepsilon;$$

$$|\bar{l}_3(N)| \leq \sup |l(Z)| \sum_{i \in I, j \in J} \left| \frac{1}{N} \sum_{n=1}^N \mathbf{1}_{Z_{ij}}(\theta_n, u_n) - \frac{1}{N^2} \sum_{n_1=1}^N \mathbf{1}_{X_i}(\theta_{n_1}) \sum_{n_2=1}^N \mathbf{1}_{Y_j}(u_{n_2}) \right|$$

$\rightarrow 0 \quad (N \rightarrow \infty);$

$$|\bar{l}_4(N)| \leq \text{diam}(l(Z_0))$$

$$\times \left[ \sum_{(i,j) \in E_0} \left[ \frac{1}{N^2} \sum_{n_1=1}^N \mathbf{1}_{X_i}(\theta_{n_1}) \times \sum_{n_2=1}^N \mathbf{1}_{Y_j}(u_{n_2}) - \frac{1}{N} \sum_{n=1}^N \mathbf{1}_{Z_{ij}}(\theta_n, u_n) \right] \right.$$

$$\left. + \sum_{(i,j) \in E_0} \frac{1}{N} \sum_{n=1}^N \mathbf{1}_{Z_{ij}}(\theta_n, u_n) \right].$$

Therefore

$$\limsup_{N \rightarrow \infty} |\bar{l}_4(N)| \leq \varepsilon \text{diam}(l(Z)).$$

Finally,

$$|\bar{l}_5(n)| \leq \varepsilon.$$

Thus, by virtue of the arbitrariness of  $\varepsilon > 0$ ,

$$L(\theta, u) = \limsup_{N \rightarrow \infty} \bar{l}(N) \leq \limsup_{N \rightarrow \infty} \bar{l}_6(N).$$

But

$$\begin{aligned}
 \bar{l}_6(N) &= \sum_{\substack{i \in I \\ j \in J}} \frac{1}{N^2} \sum_{n_1=1}^N \mathbf{1}_{X_i}(\theta_{n_1}) \sum_{n_2=1}^N \mathbf{1}_{Y_j}(u_{n_2}) l(x_i, u_{n_2}) \\
 &\leq \frac{1}{N} \sum_{n_1=1}^N \sum_{i \in I} \mathbf{1}_{X_i}(\theta_{n_1}) \sup_{x \in X} \frac{1}{N} \sum_{n_2=1}^N \sum_{j \in J} \mathbf{1}_{Y_j}(u_{n_2}) l(x, u_{n_2}) \\
 &= \sup_{x \in X} \frac{1}{N} \sum_{n_2=1}^N l(x, u_{n_2}),
 \end{aligned}$$

and hence

$$\begin{aligned}
 L(\theta, u) &\leq \limsup_{N \rightarrow \infty} \sup_{x \in X} \frac{1}{N} \sum_{n=1}^N l(x, u_n) \\
 &= \lim_{N_1 \rightarrow \infty} \sup_{N > N_1} \sup_{x \in X} \frac{1}{N} \sum_{n=1}^N l(x, u_n) \\
 &= \lim_{N_1 \rightarrow \infty} \sup_{x \in X} \sup_{N > N_1} \frac{1}{N} \sum_{n=1}^N l(x, u_n).
 \end{aligned}$$

It remains to show that the last expression does not exceed the value

$$\sup_{x \in X} \lim_{N_1 \rightarrow \infty} \sup_{N > N_1} \frac{1}{N} \sum_{n=1}^N l(x, u) = \sup_{x \in X} \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N l(x, u_n).$$

Note that otherwise,  $N_0 \in \mathbb{N}$  could be found such that for all  $N_1 \geq N_0$ ,

$$\sup_{x \in X} \sup_{N > N_1} \frac{1}{N} \sum_{n=1}^N l(x, u_n) > \sup_{x \in X} \lim_{N_2 \rightarrow \infty} \sup_{N > N_2} \frac{1}{N} \sum_{n=1}^N l(x, u_n),$$

and this means that for some  $x_1 \in X$ , we have

$$\sup_{N > N_1} \frac{1}{N} \sum_{n=1}^N l(x_1, u_n) > \sup_{x \in X} \lim_{N_2 \rightarrow \infty} \sup_{N > N_1} \frac{1}{N} \sum_{n=1}^N l(x, u_n) \geq \lim_{N_2 \rightarrow \infty} \sup_{N > N_2} \frac{1}{N} \sum_{n=1}^N l(x_1, u_n)$$

for all  $N_1 \geq N_0$ , whence a contradiction follows from

$$\lim_{N_1 \rightarrow \infty} \sup_{N > N_1} \frac{1}{N} \sum_{n=1}^N l(x_1, u_n) > \lim_{N_2 \rightarrow \infty} \sup_{N > N_2} \frac{1}{N} \sum_{n=1}^N l(x_1, u_n).$$

Theorem 3.3 is proved.

One can interpret this theorem in the following manner: when  $\theta$  changes or behaves “independently,” then for the purposes of control it is the same as if  $\theta$  were a stationary sequence.

### 3.4 Risk Estimates

We shall call the value

$$\inf_{u \in U} \mathcal{K}_l(u),$$

i.e., the lower bound of guaranteed average losses under  $\Gamma_l$ -indifference, *risk* and denote it by  $\rho(\Gamma_l)$ .

**Theorem 3.4.** *Let  $p \in \mathcal{P}(X)$ ,  $q \in \mathcal{P}(Y)$ , and*

$$\inf_{G \in \mathbf{G}} \sup_{y \in Y} p\{x' \in X : (x', y) \in G\} = \inf_{G \in \mathbf{G}} \sup_{x \in X} q\{y' \in Y : (x, y') \in G\} = 0.$$

*Then the inequalities*

$$\inf_{y \in Y} \int l(x, y) p(dx) \leq \rho(\Gamma_l) \leq \sup_{x \in X} \int l(x, y) q(dy) \quad (3.6)$$

*make sense and are correct.*

*Proof.* Let  $\mathcal{A}$  and  $\mathcal{B}$  be the algebras of sets generated by the families  $\mathbf{A}$  and  $\mathbf{B}$  respectively. It is clear that  $\mathcal{A}$  and  $\mathcal{B}$  contain no more than a countable number of elements. Define

$$\Theta_p = \left\{ \theta \in \Theta : \forall A \in \mathcal{A} \left( \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \mathbf{1}_A(\theta_n) = p(A) \right) \right\},$$

$$U_q = \left\{ u \in U : \forall B \in \mathcal{B} \left( \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \mathbf{1}_B(u_n) = q(B) \right) \right\}$$

$$\forall p \in \mathcal{P}(X), \quad q \in \mathcal{P}(Y).$$

It is clear that  $\Theta_p \neq \emptyset$ , since the sequence of independent random elements with appropriate distribution belongs to this family with probability 1. Similarly,  $U_q \neq \emptyset$ .

Choose an arbitrary  $\varepsilon > 0$  and a set  $G_\varepsilon \in \mathbf{G}$  such that

$$\max \left\{ \sup_{y \in Y} p \{x' \in X : (x', y) \in G_\varepsilon\}, \sup_{x \in X} q \{y' \in Y : (x, y') \in G_\varepsilon\} < \varepsilon \right\}.$$

Now, having fixed an arbitrary point  $x \in X$  and an arbitrary sequence  $u \in U_q$ , set

$$G^x = \{y \in Y : (x, y) \in G_\varepsilon\}.$$

It is easy to see that  $G^x$  is an open set. Indeed, that  $G_\varepsilon$  is open means that it is a union of some open neighborhoods of all its points, and one can choose these neighborhoods in the form

$$D_z = A_x \times B_y \quad (z \in G_\varepsilon, z = (x, y)), \quad x \in A_x \in \mathbf{A}, y \in B_y \in \mathbf{B}.$$

Furthermore,

$$G^x = \bigcup_{y \in G^x} B_y,$$

i.e.,  $G^x$  is open. Hence the set

$$F^x = Y \setminus G^x$$

is compact. Let us surround every point  $y \in F^x$  by an open neighborhood  $B_y$  such that

$$\text{diam}(l(x, B_y)) = \sup_{y', y'' \in B_y} |l(x, y') - l(x, y'')| < \varepsilon,$$

and then, from this open cover of the set  $F^x$ , choose a finite subcover  $\{B_1, B_2, \dots, B_k\}$ , and then assume that

$$Y_0 = Y \setminus \bigcup_{s'=1}^k B_{s'}, \quad Y_s = B_s \setminus \bigcup_{1 \leq s' \leq s} B_{s'} \quad (1 \leq s \leq k).$$

Obviously,

$$Y = \sum_{s=0}^k Y_s, \quad Y, Y_s \in \mathcal{B}; \quad \text{diam } l(x, Y_s) < \varepsilon \quad (1 \leq s \leq k),$$

$$q(Y_0) < q(G^x) < \varepsilon.$$

Without loss of generality, assuming that

$$Y_s \neq \emptyset \quad (0 \leq s \leq k),$$

choose the points

$$y_s \in Y_s \quad (0 \leq s \leq k),$$

one for each  $Y_s$ , and express

$$\bar{l}(N) = \frac{1}{N} \sum_{n=1}^N l(x, u_n)$$

as the sum

$$\bar{l}(N) = \frac{1}{N} \sum_{m=1}^3 \bar{l}_m(N),$$

where

$$\bar{l}_1(N) = \sum_{s=1}^k \frac{1}{N} \sum_{n=1}^N [l(x, u_n) - l(x, y_s)] \mathbf{1}_{Y_s}(u_n);$$

$$\bar{l}_2(N) = \frac{1}{N} \sum_{n=1}^N [l(x, u_n) - l(x, y_0)] \mathbf{1}_{Y_0}(u_n);$$

$$\bar{l}_3(N) = \sum_{s=0}^k l(x, y_s) \frac{1}{N} \sum_{n=1}^N \mathbf{1}_{Y_s}(u_n).$$

We have

$$|\bar{l}_1(N)| < \varepsilon \sum_{s=1}^k \frac{1}{N} \sum_{n=1}^N \mathbf{1}_{Y_s}(u_n) \leq \varepsilon;$$

$$|\bar{l}_2(N)| < \text{diam}(l(X \times Y)) \frac{1}{N} \sum_{n=1}^N \mathbf{1}_{Y_0}(u_n) \xrightarrow{N \rightarrow \infty} \text{diam}(l(X \times Y))q(Y_0)$$

$$< \varepsilon \text{diam}(l(X \times Y));$$

$$\bar{l}_3(N) \xrightarrow{N \rightarrow \infty} \sum_{s=0}^k l(x, y_s)q(Y_s);$$

and

$$\left| \sum_{s=1}^k l(x, y_s)q(Y_s) - \int_{\bigcup_{s=1}^k Y_s} l(x, y)q(dy) \right| \leq \varepsilon;$$

$$\left| l(x, y_0)q(y_0) - \int_{Y_0} l(x, y)q(dy) \right| \leq \text{diam} l(X \times Y) \times \varepsilon;$$

and in view of the arbitrariness of  $\varepsilon > 0$ ,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N l(x, u_n) = \int l(x, y) q(dy).$$

But in this case it follows from Theorem 3.3 that

$$\mathcal{K}_l(u) = \sup_{x \in X} \limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N l(x, u_n) = \sup_{x \in X} \int l(x, y) q(dy),$$

from which we obtain the right-hand inequality (3.6).

In order to prove the left-hand inequality in (3.6), notice that if instead of the triple  $(X, Y, l)$  we consider the triple  $(X, Y, l')$ , where

$$l' : Y \times X \rightarrow \mathbb{R}, \quad l'(y, x) = -l(x, y),$$

then when  $(u, \theta) \in \Gamma_l$ , due to the inequality 3.5, we have

$$\begin{aligned} -\liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N l(\theta_n, u_n) &= \limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N l'(u_n, \theta) \\ &\leq \sup_{y \in Y} \limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N l'(y, \theta_n) \\ &= -\inf_{y \in Y} \liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N l(\theta_n, y), \end{aligned}$$

i.e.,

$$\liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N l(\theta_n, u_n) \geq \inf_{y \in Y} \liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N l(\theta_n, y).$$

Now to every sequence  $u \in U$  associate some sequence

$$\theta^u \in \Theta_p \cap \Gamma_l(u).$$

This can be done due to Theorem 3.2 and the countability of the algebra of sets  $\mathcal{A}$ . We obtain

$$\inf_{u \in U} \sup_{\theta \in \Gamma_l(u)} \limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N l(\theta_n, u_n) \geq \inf_{u \in U} \limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N l((\theta_n^u), u_n).$$

But

$$\begin{aligned} \limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N l((\theta_n^u), u_n) &\geq \liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N l((\theta_n^u), u_n) \\ &\geq \inf_{y \in Y} \liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N l((\theta_n^u), y), \end{aligned}$$

and since  $\theta^u \in \Theta_p$ , one can prove, as was done above, that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N l((\theta_n^u), y) = \int_X l(x, y) p(dx).$$

We therefore obtain that

$$\rho(\Gamma_l) \geq \inf_{y \in Y} \int l(x, y) p(dx).$$

The proof of Theorem 3.4 is complete.

A stronger statement can be obtained in the case that  $X$  and  $Y$  are complete separable metric spaces.

**Theorem 3.5.** *If  $X$  and  $Y$  are complete separable metric spaces and  $l$  is a Borel function, then for any*

$$p \in \mathcal{P}(X), \quad q \in \mathcal{P}(Y),$$

*the following inequalities make sense and are correct:*

$$\inf_{y \in Y} \int l(x, y) p(dx) \leq \rho(\Gamma_l) \leq \sup_{x \in X} \int l(x, y) q(dy). \quad (3.7)$$

*Proof.* We show first of all that cross sections  $l(x, \cdot)$  are Borel functions for all  $x \in X$ . For this purpose, we note that if

$$r \in \mathbb{R}, \quad F_r = \{(x, y) : l(x, y) \leq r\},$$

then

$$F_r \in \mathcal{A} \otimes \mathcal{B},$$

since  $l$  is a Borel function. In such a case, in accordance with [23, Lemma 2], we have

$$Y_{xr} = \{y \in Y : (x, y) \in F_r\} \in \mathcal{B},$$

for all  $x \in X$  as a cross section of a measurable set, and hence  $l(x, \cdot)$  is a Borel function. Precisely the same proof is valid for the function  $l(\cdot, y)$  for all  $y \in Y$ . So we have proved that the required inequalities (3.7) make sense.

Now we use Luzin's theorem in the following form [71, Corollary 24.22]:

**Theorem 3.6 (Luzin).** *Let  $X$  and  $Y$  be complete separable metric spaces and let  $\mu$  be a probability measure on the  $\sigma$ -algebra  $\mathcal{B}$  of Borel sets of space  $X$ . If  $f$  is a Borel mapping from  $X$  into  $Y$ , then for any  $\varepsilon > 0$  there is a compact set  $K_\varepsilon \subset X$  such that  $\mu(K_\varepsilon) > 1 - \varepsilon$  and the narrowing of  $f$  on  $K_\varepsilon$  is continuous.*

Since in our case  $Y$  and  $\mathbb{R}$  are complete separable metric spaces,  $q$  is the probability measure on the  $\sigma$ -algebra  $\mathcal{B}$  of Borel sets of the space  $Y$ , and  $l(x, \cdot)$  is a Borel mapping from  $Y$  into  $\mathbb{R}$ , then for any  $\varepsilon > 0$  there is a compact set  $F^\varepsilon \subset Y$  such that  $q(F^\varepsilon) > 1 - \varepsilon$  and the narrowing  $l(x, \cdot)$  on  $F^\varepsilon$  is continuous.

The rest of the proof is carried out in exactly the same way as for Theorem 3.4. Theorem 3.5 is thus proved.

### 3.5 Decision-Making under $\Gamma_I$ -independence and Antagonistic Games

Let us suppose that the function  $l$  is measurable relative to the  $\sigma$ -algebra  $\mathcal{A} \otimes \mathcal{B}$  and define

$$\mathfrak{L} : \mathcal{P}(X) \times \mathcal{P}(Y) \rightarrow \mathbb{R}, \quad \mathfrak{L}(p, q) = \int l(x, y) p(dx) q(dy),$$

and

$$\bar{v} = \inf_{q \in \mathcal{P}(Y)} \sup_{p \in \mathcal{P}(X)} \mathfrak{L}(p, q), \quad \underline{v} = \sup_{p \in \mathcal{P}(X)} \inf_{q \in \mathcal{P}(Y)} \mathfrak{L}(p, q).$$

Thus  $\bar{v}$  and  $\underline{v}$  are respectively the upper and the lower values of the antagonistic game  $(\mathcal{P}(X), \mathcal{P}(Y), \mathfrak{L})$ .

**Theorem 3.7.** *If the conditions of Theorem 3.4 are satisfied and if there exist  $p \in \mathcal{P}(X)$  and  $q \in \mathcal{P}(Y)$  such that*

$$\inf_{y \in Y} \int l(x, y) p(dx) = \sup_{x \in X} \int l(x, y) q(dy) \stackrel{\text{def}}{=} v', \quad (3.8)$$

then

$$\underline{v} = \bar{v} = v' = \rho(\Gamma_I) = \min_{u \in U} \mathcal{K}_l(u) = \mathcal{K}_l(u^*) \quad (3.9)$$

for any sequence  $u^* \in U_q$ .

*Proof.* Since  $Y$  is a Hausdorff space, the set consisting of one arbitrary point  $y \in Y$  is measurable. Therefore

$$\underline{v} \geq \inf_{q' \in \mathcal{P}(Y)} \mathfrak{L}(p, q') = \inf_{y \in Y} \int l(x, y) p(dx) = v'.$$

Similarly,

$$\bar{v} \leq \sup_{p' \in \mathcal{P}(Y)} \mathfrak{L}(p', q) = \sup_{x \in X} \int l(x, y) q(dy) = v'.$$

It remains to make use of Theorem 3.4 and of the well-known fact that  $\underline{v} \leq \bar{v}$  always. Theorem 3.7 is thus proved.

**Theorem 3.8.** *If  $X$  and  $Y$  are complete separable metric spaces and the function  $l$  is a Borel function, then  $\underline{v} \leq \rho(\Gamma_l) \leq \bar{v}$ , and (3.7) follows from (3.6).*

*Proof.* According to the definition of  $\underline{v}$  and  $\bar{v}$ , for any  $\varepsilon > 0$  one can find  $p_\varepsilon \in \mathcal{P}(X)$  and  $q_\varepsilon \in \mathcal{P}(Y)$  such that

$$\inf_{y \in Y} \int l(x, y) p_\varepsilon(dx) > \underline{v} - \varepsilon, \quad \sup_{x \in X} \int l(x, y) q_\varepsilon(dy) < \bar{v} + \varepsilon.$$

But then, according to Theorem 3.5,

$$\underline{v} - \varepsilon \leq \rho(\Gamma_l) \leq \bar{v} + \varepsilon,$$

which by virtue of the arbitrariness of  $\varepsilon > 0$ , proves the required inequalities. In this case, if

$$\underline{v} = \bar{v} = \sup_{x \in X} \int l(x, y) q(dy)$$

for some  $q \in \mathcal{P}(Y)$ , we obtain (3.9), making use of the relation (3.3).

**Theorem 3.9.** *If the function  $l$  is continuous, then*

$$\underline{v} = \bar{v} = \rho(\Gamma_l) = \inf_{u \in \tilde{U}} \mathcal{K}_l(u),$$

where

$$\tilde{U} = \bigcup_{q \in \mathcal{P}(Y)} U_q.$$

*Proof.* By virtue of the compactness of the spaces  $X$  and  $Y$ , the continuity of the function  $l$ , in accordance with Glikhsberg's theorem [25], guaranties the existence of the value of the game

$$(\mathcal{P}(X), \mathcal{P}(Y), \mathfrak{L}),$$

i.e., the validity of the equality  $\underline{v} = \bar{v} = v$ . This means that for any  $\varepsilon > 0$  one can find  $p \in \mathcal{P}(X)$  and  $q \in \mathcal{P}(Y)$  such that on the one hand, the condition of

Theorem 3.4 will be satisfied (since  $C = \emptyset$ ), and on the other hand, the following inequalities will be true:

$$v - \varepsilon \leq \inf_{q' \in \mathcal{P}(Y)} \mathfrak{L}(p, q') \leq \sup_{p' \in \mathcal{P}(Y)} \mathfrak{L}(p', q) \leq v + \varepsilon.$$

From this we immediately obtain that

$$v - \varepsilon \leq \inf_{y \in Y} \int l(x, y) p(dx) \leq \sup_{x \in X} \int l(x, y) q(dy) \leq v + \varepsilon,$$

and further, making use of Theorem 3.4 and also the inequality

$$\rho(\Gamma_l) \leq v + \varepsilon,$$

which arises from the proof of that theorem, we prove the validity of Theorem 3.9.

### 3.6 Some Examples of $\Gamma_l$ -Independence

*Example 3.1.* Let us consider an example in which average losses reach their lower value [54].

**Theorem 3.10.** *Let  $X = Y = [0, 1]$  with the natural topology of a segment,*

$$l(x, y) = \begin{cases} -1, & \text{if } x < y < x + \frac{1}{2}; \\ 0, & \text{if } y = x \text{ or } y = x + \frac{1}{2}; \\ +1, & \text{in all other cases.} \end{cases}$$

Then

$$\rho(\Gamma_l) = \min_{u \in U} \mathcal{K}_l(u) = \underline{v} < \bar{v}.$$

*Proof.* The corresponding game  $(\mathcal{P}(X), \mathcal{P}(Y), \mathfrak{L})$  was proposed in [74] as a game that does not have a value. The authors showed that in this case,  $\underline{v} = \frac{1}{3}$ ,  $\bar{v} = \frac{3}{7}$ . Our goal is to construct a sequence  $u^* \in U$  such that  $\mathcal{K}_l(u^*) = \frac{1}{3}$ . Setting

$$u_n^* = \begin{cases} \frac{1}{2} - \frac{1}{n}, & \text{if } n = 3k, \quad k \in N, \\ 1, & \text{in other cases,} \end{cases}$$

and writing for short  $l(x, u_n^*) = f_n(x)$ , we consider four possible cases:

(1)  $x = 1$ ,

$$f_n(x) = \begin{cases} 1, & n = 3k, \\ 0, & n \neq 3k; \end{cases}$$

$$(2) x = \frac{1}{2},$$

$$f_n(x) = \begin{cases} 0, & n = 3k, \\ 1, & n \neq 3k; \end{cases}$$

$$(3) 0 \leq x < \frac{1}{2},$$

$$f_n(x) = \begin{cases} -1, & n = 3k, \\ 1, & n \neq 3k; \end{cases}$$

$$(4) \frac{1}{2} < x < 1,$$

$$f_n(x) = \begin{cases} 1, & n = 3k, \\ -1, & n \neq 3k. \end{cases}$$

Now make use of the well-known fact that if  $f_n \rightarrow f$ , then

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f_n = f.$$

In our case

$$\begin{aligned} f(x) &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f_n(x) = \lim_{N \rightarrow \infty} \frac{1}{N} \left[ \sum_{\substack{n \leq N \\ n=3k}} f_n(x) + \sum_{\substack{n \leq N \\ n \neq 3k}} f_n(x) \right] \\ &= \frac{1}{3} \lim_{\substack{n \rightarrow \infty \\ n=3k}} f_n(x) + \frac{2}{3} \lim_{\substack{n \rightarrow \infty \\ n \neq 3k}} f_n(x), \end{aligned}$$

and therefore

$$f(x) = \begin{cases} \frac{1}{3} \cdot 1 + \frac{2}{3} \cdot 0 = \frac{1}{3}, & \text{if } x = 1 \text{ or } x = \frac{1}{2}; \\ \frac{1}{3} \cdot (-1) + \frac{2}{3} \cdot 1 = \frac{1}{3}, & \text{if } 0 \leq x < \frac{1}{2}; \\ \frac{1}{3} \cdot 1 + \frac{2}{3} \cdot (-1) = -\frac{1}{3}, & \text{if } \frac{1}{2} < x < 1. \end{cases}$$

On the basis of Theorem 3.3 one can conclude that

$$\mathcal{H}_1(u^*) = \max \left\{ \frac{1}{3}, -\frac{1}{3} \right\} = \frac{1}{3}.$$

Theorem 3.10 is proved.

*Remark 3.1.* It is worth noticing that one can obtain the same result by making use not of sequences, but of finitely additive mixed strategies [87]. However, these two approaches, although they look essentially similar, cannot be interchanged, since not every finitely additive measure corresponds to some sequence, and not every sequence corresponds to some finitely additive measure. This circle of questions is discussed in detail in Chapter 4.

*Example 3.2.* We return to the example discussed at the beginning of Section 3.1. First we shall prove the following theorem.

**Theorem 3.11.** *Let  $X = Y = [0, 1]$  with the natural topology of the segment  $0 < \delta < 1$ , and let*

$$l(x, y) = \begin{cases} 0, & \text{if } |x - y| \leq \delta, \\ 1, & \text{if } |x - y| > \delta, \end{cases}$$

and

$$m = \frac{1}{2\delta}.$$

Then

$$\min_{u \in U} \mathcal{K}_l(u) = \begin{cases} 1 - \frac{1}{[m]}, & \text{if } [m] = m, \\ 1 - \frac{1}{[m]+1}, & \text{if } [m] < m, \end{cases} \quad (3.10)$$

where  $[m]$  is the integral part of the number  $m$ .

*Remark 3.2.* To prove this theorem one can use general methods of game theory [83], but it is more interesting to give a straightforward proof, based only on the results mentioned above.

*Proof.* Set  $r = m - [m]$ . The proof of the theorem will depend on which of the following three possible cases takes place:

- (1)  $r = 0$ ,
- (2)  $0 < r \leq \frac{1}{2}$ ,
- (3)  $\frac{1}{2} < r \leq 1$ .

We consider them in turn.

Case  $r = 0$ . Set  $x_k = (2k - 1/k)\delta$ ,  $y_k = (2k - 1)\delta$ ,  $p(x_k) = q(y_k) = \frac{1}{m}$ ,  $k = 1, 2, \dots, m$ . Obviously,  $0 < x_k < 1$  and

$$x_{k+1} - x_k = [(2(k+1) - 1/k + 1) - (2k - 1/k)]\delta = [2 + 1/k - 1/k + 1]\delta > 2\delta,$$

so for any  $y \in [0, 1]$  there exists no more than one point  $x_k$  at a distance from  $y$  not exceeding  $\delta$ , and therefore

$$\int l(x, y)p(dx) = \begin{cases} 1, & \text{if there are no such points,} \\ 1 - \frac{1}{m}, & \text{if there is one such point.} \end{cases}$$

On the other hand, obviously,  $0 < y_k < 1$  and

$$y_{k+1} - y_k = [(2(k+1) - 1) - (2k - 1)]\delta = 2\delta,$$

so for any  $x \in [0, 1]$  there exist either one or two points  $y_k$  at a distance not exceeding  $\delta$  from  $x$ , and therefore

$$\int l(x, y)q(dy) = \begin{cases} 1 - \frac{1}{m}, & \text{if there is one such point,} \\ 1 - \frac{1}{2m}, & \text{if there are two such points,} \end{cases}$$

i.e.,

$$\sup_{x \in X} \int l(x, y)q(dy) = 1 - \frac{1}{m}.$$

From this, on the basis of Theorem 3.8, we conclude that

$$\min_{u \in U} \mathcal{K}_l(u) = 1 - \frac{1}{m},$$

and it is attained by any sequence  $u^* \in U_q$ .

Case  $0 < r \leq \frac{1}{2}$ . Set  $a_k = 2(k-1)\delta$  and  $b_k = a_k + 2r\delta$   $k = 1, 2, \dots, [m] + 1$ . Obviously,

$$0 = a_1 \leq a_k \leq a_{[m]+1} = 2[m]\delta < 1,$$

$$0 < b \leq b_k \leq b_{[m]+1} = a_{[m]+1} + 2r\delta = 2[m]\delta + 2r\delta = 2\delta m = 1,$$

$$a_{k+1} - a_k = 2\delta.$$

Thus, the segments  $[a_k, b_k]$  have the same length, equal to  $2r\delta$ , and so less than  $\delta$ , and they are spaced with the “step”  $2\delta$  on the segment  $[0, 1]$ . Denote their union by  $A$ :

$$A = \sum_{k=1}^{[m]+1} [a_k, b_k].$$

It is evident that for any given point  $x \in [0, 1]$ , the set of points  $y \in A$  situated at a distance less than  $\delta$  from  $x$  is either one of segments  $[a_k, b_k]$  or the union of the two parts of two adjacent segments with the total length of these parts equal to the length of one segment. Therefore, if  $q$  is the uniform distribution on  $A$ , then

$$\int l(x, y)q(dy) = 1 - 1/[m] + 1 \quad \forall x \in X.$$

Now let

$$x_k = 2(k-1) \left(1 + \frac{r}{[m]}\right) \delta \quad (k = 1, 2, \dots, [m] + 1).$$

It is evident that

$$0 = x_1 \leq x_k \leq x_{[m]+1} = 2\delta[m] \left(1 + \frac{r}{[m]}\right) = 2\delta([m] + r) = 2\delta m = 1.$$

This is a collection of  $[m] + 1$  points, situated at distance

$$2\delta \left(1 + \frac{r}{[m]}\right) > 2\delta$$

from one another, and hence no more than one point  $x_k$  can be found in the neighborhood of radius  $\delta$  of any given point  $y \in Y$ . This means that when

$$p(x_k) = \frac{1}{[m] + 1},$$

we have

$$\int l(x, y)p(dx) = \begin{cases} 1 - \frac{1}{[m]+1}, & \text{if there is one such point,} \\ 1, & \text{if there are no such point.} \end{cases}$$

Thus

$$\inf_{y \in Y} \int l(x, y)p(dx) = 1 - \frac{1}{[m] + 1}.$$

Again making use of Theorem 3.8, we obtain that

$$\min_{u \in U} \mathcal{K}_l(u) = 1 - \frac{1}{[m] + 1},$$

and it is attained on any sequence  $u^* \in U_q$ .

Case  $\frac{1}{2} < r < 1$ . Set

$$b_k = (2k-1)\delta, \quad a_k = b_k - 2\delta(1-r) \quad (k = 1, 2, \dots, [m] + 1).$$

Obviously,

$$\begin{aligned}
0 < \delta = b_1 \leq b_k \leq b_{[m]+1} &= [2([m] + 1) - 1]\delta = (2[m] + 1)\delta \\
&= 2\delta \left( [m] + \frac{1}{2} \right) < 1, \\
0 < \delta - 2\delta(1 - r) = a_1 \leq a_k \leq a_{[m]+1} &= 2\delta \left( [m] + \frac{1}{2} \right) - 2\delta(1 - r) \\
&= 2\delta \left( [m] + r - \frac{1}{2} \right) = 1 - \delta, \\
b_{k+1} - b_k &= 2\delta.
\end{aligned}$$

Thus the segments  $[a_k, b_k]$  have the same length, equal to  $2\delta(1 - r)$ , i.e., less than  $\delta$ , and are placed on the segment  $[0, 1]$  with the “step”  $2\delta$ . Setting

$$A = \sum_{k=1}^{[m]+1} [a_k, b_k],$$

we see that for any point  $x \in [0, 1]$ , the intersection of the set  $A$  with the  $\delta$ -neighborhood of the point  $x$  is either one of the segments  $[a_k, b_k]$  entirely, or two parts of two adjacent segments with total length equal to the length of one segment.

And again, denoting by  $q$  the uniform distribution on  $A$ , we obtain

$$\int l(x, y)q(dx) = \frac{1}{[m] + 1}$$

for any  $x \in X$ . But choosing the measure  $p \in \mathcal{P}(X)$  in the same way as in the previous case, that is,

$$p(x_k) = \frac{1}{[m] + 1} \quad \text{for } k = 1, \dots, [m] + 1,$$

we make relation (3.8) valid, and thus, making use of Theorem 3.8, obtain the required equality for this case. Theorem 3.11 is proved.

Now speaking formally, in order to find the solution of the problem discussed at the beginning of Section 3.1 it is sufficient to choose the value  $[t_2 - t_1]$  as the unit of time measurement, and to apply the proved Theorem 3.11 when  $\delta = \Delta t / (t_2 - t_1)$ . It is clear, however, that the formula (3.10), which depends on whether the number  $(t_2 - t_1) / 2\Delta t$  is an integer, is of no practical use. At the same time, construction of the optimal sequence realizing the precise value of  $\min_{u \in U} \mathcal{K}_1(u)$  is too complicated. However, accepting the obtained value as a benchmark, it is easy to construct a suboptimal algorithm and to estimate its error.

Let us break up the segment  $[t_1, t_2]$  into equal parts by the points

$$t_1 = \tau_0, \tau_1, \dots, \tau_n = t_2, \quad \tau_{i+1} - \tau_i = (t_2 - t_1)/n$$

and choose in turn  $\bar{t}_1 = \tau_0, \bar{t}_2 = \tau_1, \dots, \bar{t}_n = \tau_{n-1}, \bar{t}_{n+1} = \tau_n, \bar{t}_{n+2} = \tau_0, \bar{t}_{n+3} = \tau_1$ , etc.

Intuitively, this is clear, and it corresponds exactly to the formal definition of  $\Gamma$ -independence that the sequence  $\bar{t}^*$  will not “spy” on the sequence  $\bar{t}$ . That is, on the basis of Theorem 3.4 we have

$$\frac{n}{n+1} \left( 1 - \frac{\Delta t}{t_2 - t_1} \right) \leq \mathcal{H}_l(\bar{t}) \leq \frac{n}{n+1} \left( 1 - \frac{\Delta t}{t_2 - t_1} \right) + \frac{1}{m+1},$$

or for sufficiently large  $n$ ,

$$\mathcal{H}_l(\bar{t}) \approx 1 - \frac{\Delta t}{t_2 - t_1}.$$

*Example 3.3.* Let  $X = Y = (1, 2, \dots, m)$ . The loss function has the form

$$l(x, y) = \begin{cases} 1, & \text{if } x = y, \\ 0, & \text{if } x \neq y, \end{cases} \quad x \in X, y \in Y.$$

Thus in order to eliminate losses, the control must avoid coincidence with the unknown parameter.

According to Theorem 3.3,

$$\mathcal{H}_l(u) = \max_{x \in X} \limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N l(x, y_n) \quad \forall u \in U.$$

This allows us to reduce the control problem to a finite antagonistic game. Denote by  $v$  the value of the game  $(X, Y, l)$ , i.e., set

$$v = \max_{p \in \mathcal{P}(X)} \min_{q \in \mathcal{P}(Y)} \sum_{\substack{x \in X \\ y \in Y}} l(x, y) p(x) q(y),$$

or in view of the finiteness of the sets  $X$  and  $Y$ ,

$$v = \min_{q \in \mathcal{P}(Y)} \max_{p \in \mathcal{P}(X)} \sum_{\substack{x \in X \\ y \in Y}} l(x, y) p(x) q(y),$$

where  $\mathcal{P}(\cdot)$  are all possible distributions on  $(\cdot)$ .

Let  $q^* \in \mathcal{P}(Y)$  be one of those distributions (mixed strategies) on which the appropriate minimum is attained:

$$\min_{q \in \mathcal{P}(Y)} \max_{p \in \mathcal{P}(X)} \sum_{\substack{x \in X \\ y \in Y}} l(x, y) p(x) q(y) = v = \max_{p \in \mathcal{P}(X)} \sum_{\substack{x \in X \\ y \in Y}} l(x, y) p(x) q^*(y).$$

On the basis of Theorem 3.9 we have in this case

$$\min_{u \in U} \mathcal{K}_l(u) = \mathcal{K}_l(u^*) = v \quad \forall u^* \in U^*,$$

$$U^* \in \left\{ u \in U : \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \mathbf{1}_Y(y_n) = q^*(y) \quad \forall y \in Y \right\}.$$

Now taking into account the form of loss function, it is obvious that the uniform distribution upon  $Y$  will be optimal, i.e.,  $q^* = 1/m$  for all  $y \in Y$ , and

$$\min_{u \in U} \mathcal{K}_l(u) = \mathcal{K}_l(u^*) = v = \frac{1}{m} \quad \forall u^* \in U^*.$$

According to this, the sequence

$$u^* = \{1, 2, \dots, m, 1, 2, \dots, m, 1, 2, \dots\}$$

in particular will be optimal. It is interesting to note that if we did not consider the available information on independence between  $\theta$  and  $u$  in this example, i.e., if under the same conditions we solved the problem

$$\sup_{\theta \in \Theta} L(\theta, u) \rightarrow \inf, \quad u \in U,$$

then we would have been able to provide the guaranteed level of the average losses equal to 1 instead of  $\frac{1}{m}$ , and we would have no reason to prefer one control to another.

*Example 3.4.* Now with the same  $X = Y = (1, 2, \dots, m)$ , let the result of the operation be evaluated by the loss function

$$l(x, y) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x \neq y. \end{cases}$$

That is, to eliminate losses the control must coincide with the unknown parameter. Then obviously,

$$\min_{u \in U} \mathcal{K}_l(u) = \mathcal{K}_l(u^*) = v = 1 - \frac{1}{m} \quad \forall u^* \in U^*.$$

Comparing this example with the previous one, we note that in this case the optimal guaranteed value of average losses tends to 1 as  $m$  increases. These two variants of loss function in some sense are polar opposites. In the interval between them is a set of other loss functions that can occur in practice.

## Chapter 4

# Nonstochastic Randomness

Si, lorsque le nombre des épreuves est très grand, cette fréquence ne tend pas vers une limite, mais oscille plus ou moins lentement entre diverses limites, on devra affirmer que la probabilité  $p$  ne reste pas constante, mais varie au cours des épreuves.

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Émile Borel

Speaking of randomness in the ordinary sense of this word, we mean those phenomena in which we do not find regularities allowing us to predict their behavior. Generally speaking, there are no reasons to assume that random in this sense phenomena are subject to some probabilistic laws. Hence, it is necessary to distinguish the randomness in this *broad sense* and stochastic randomness (which is the subject of probability theory).

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A. N. Kolmogorov

### 4.1 Preliminaries

The results of Chapter 3 allow us to look in a new way at the problem of decision-making under uncertainty as a whole. Let us begin by noticing that in research in this field there is one at first sight terminological, but in fact rather fundamental, confusion. Some authors, following [46], mention only two types of behavior of the cause–effect mechanism that generates consequences: (1) so-called complete uncertainty, or just uncertainty, when nothing is known about the mechanism and (2) so-called risk, when consequences are random with a given probability distribution. Others, such as [80] and [79], consider this second variant as one of the main types of uncertainty, and call it correspondingly *stochastic uncertainty* or a *first information situation*.

This terminological diversity has deep roots. For applications it is quite natural to understand any ambiguity with respect to the unknown parameter as un-

certainty. At the same time, for pure mathematics it is much more important that methods and approaches to a decision-making problem under stochastic behavior of the parameter relate to an entirely different mathematical discipline, namely probability theory, and have, it would seem, nothing in common with the rest of decision theory.

Even in [22], one of the main goals of which is promotion of a guaranteed result principle as the “only one . . . that an operations researcher can rightfully use” (p. 70), the author, speaking of stochastic uncertainty (p. 77), refuses to follow this principle strictly. If, however, we are to take advantage of the technique we introduced in Chapter 3 and consider explicitly the sequences of values of the unknown and control parameters, then we will see that in this approach stochastic uncertainty does not substantially differ from two others types:

- (1) “strict uncertainty” when one knows nothing about  $\theta$ ;
- (2) “indifferent uncertainty” when one knows only that  $\theta$  does not depend on  $u$ .

In all these cases we solve the problem

$$\sup_{\theta} \limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N l(\theta_n, u_n) \rightarrow \inf,$$

and the difference consists only in the choice of the set of possible values of the parameter  $\theta$ . In case of “full uncertainty” it is the set  $X^{\mathbb{N}}$ , while in case of “indifferent uncertainty” it is the set  $\Gamma_l(u)$ , and in case of “stochastic uncertainty” it is the set of all sequences  $\theta \in \Gamma_l(u)$  that satisfy an additional condition: the relation

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \mathbf{1}_{X_i}(\theta_n) = P(X_i)$$

is satisfied for any subsets  $X_i$  appearing in Definition 3.1 (it is assumed that  $P$  is a known distribution of the unknown parameter and all  $X_i$  are chosen as measurable). It is easy to prove that a sequence of values of independent random elements with distribution  $P$  will satisfy this condition with probability 1.

Indeed, assume that  $(X, \mathcal{A}, P)$  is a probability space and it is known that the unknown parameter is a random element with distribution  $P$ . Moreover, the set of admissible decisions  $Y$  and a bounded loss function  $l : X \times Y \rightarrow \mathbb{R}$  with measurable cross sections  $l(\cdot, y)$ ,  $y \in Y$ , are given. It is clear that the usual endeavor to minimize average losses by means of choice of  $y \in Y$  in these conditions leads us to the problem

$$\int l(x, y) P(dx) \rightarrow \inf \quad (y \in Y).$$

Now in the same way as in Chapter 3, we introduce the sets  $\Theta = X^{\mathbb{N}}$ ,  $U = Y^{\mathbb{N}}$ , and the function  $L : \Theta \times U \rightarrow \mathbb{R}$ ,

$$L(\theta, u) = \limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N l(\theta_n, u_n).$$

We add to Definition 3.1 one further condition:

$$\lim \frac{1}{N} \sum_{n=1}^N \mathbf{1}_{X_i}(\theta_n) = P(X_i) \quad \forall i \in I$$

(implying that for all  $i$ ,  $X_i \in \mathcal{A}$ ) and denote by  $\Gamma_p$  the set of all pairs  $(u, \theta) \in U \times \Theta$  that satisfy Definition 3.1 with this additional condition. From the results of Chapter 3 (see the proof of Theorem 3.4) it follows that for any  $u \in U$ ,  $\theta \in \Gamma_p(u)$ , the relation

$$L(\theta, u) = \limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \int l(x, u_n) P(dx)$$

holds, which means that

$$\inf_{u \in U} \sup_{\theta \in \Gamma_p(u)} L(\theta, u) = \inf_{y \in Y} \int l(x, y) P(dx).$$

It remains to note that if the parameter  $\theta$  is random with distribution  $P$ , then for natural assumptions,  $\theta \in \Gamma_p(u)$  with probability 1.

So if in case of stochastic (or Bayesian) uncertainty one explicitly considers infinite numbers of “trials,” the exclusiveness of this kind of uncertainty will disappear, i.e., the point is not in the specific form of information about the parameter  $\theta$ , but only in the way we are going to estimate the whole set of our decisions: all of them on average or each of them separately. In other words, it is more important to distinguish between the problems of making “mass” and “unique” decisions than to distinguish stochastic and nonstochastic problems. The “pasting together” of the “mass” property and the stochasticity is stipulated only by the absence of mathematical methods of description of the mass phenomena that are not stochastic.

Therefore Chapter 4 is dedicated to the development of mathematical tools for describing mass phenomena without any assumptions about their stochasticity [39, 43].

Let us concentrate further on terminology. The notion that in this context we consider natural to denote by the word “mass” is closely related to more habitual notions of “random” (not in the probabilistic sense, but in the common sense of the word) and of “statistical.” As is known, the notion of randomness that we use

in everyday life is considerably broader than the one used in probability theory. For example, in [47] the following is said about this:

Speaking of randomness in the ordinary sense of this word, we mean those phenomena in which we do not find regularities, allowing us to predict their behavior. Generally speaking, there are no reasons to assume that random in this sense phenomena are subjected to some probabilistic laws. Hence, it is necessary to distinguish the randomness in this *broad sense* and stochastic randomness (which is the subject of probability theory).

However, what do the words “we do not find regularities, allowing us to predict their behavior” mean? We should scarcely understand them in the sense that such regularities do not exist. Then either it is impossible to discover the unknown regularity by means of existing tools, or this regularity, for some reason, does not interest us.

The first viewpoint was proposed by A. Church in [11] and then was investigated in works of many authors (see, for example, [89] and the references therein). The second, as far as the author of this book knows, has not yet been considered. Let us consider examples of the phenomena in which the second viewpoint seems natural.

- (1) In the theory of errors [78], random errors are defined as those such that “their values change from one measurement to another.” As an example, there is a list of random errors that can appear in the measurement by a micrometer of a wire’s diameter: the error due to inaccuracy of the micrometer; the error due to deviation of the wire’s shape from cylindrical; the error due to possible ellipticity of the wire cross sections. It is clear that all these errors are subject to quite specific regularities that can be easily discovered. Since the goal of this measurement is to learn the average value of the diameter of the wire over a long piece of wire, we are not interested in local regularities.
- (2) There are many situations in which values of the variable of interest can be easily measured, but they change so rapidly that we are physically unable to react to these changes. We are compelled to consider such variables as random even when we know exactly the local regularities by which they change and to orient ourselves on some average characteristics, in other words, on global or statistical regularities.
- (3) Option investors, or even option traders, are not interested in local regularities (or explanations) of underlying price changes. A change in price of the underlying stock is a result of complex decisions of the participants in the stock market, sellers and buyers, for whom it can be vital to understand short-term market trends. For option investors and traders, on the other hand, the price of the underlying stock is a parameter about which in making their pricing decisions they are interested only in its global statistical properties.
- (4) Finally, in many cases of practical design of control systems it turns out that technically we can take into consideration local regularities, but economically this consideration is disadvantageous. It means that here we also need some coarse-grained, rough description that characterizes the phenomenon

not “in detail” but “as a whole,” i.e., again in the sense of its statistical regularities.

Therefore the following heuristic definition seems natural: mass phenomena concerning which we are interested only in their statistical regularities are called *random in a broad sense*.

An exact definition of this notion will be given later. Here we just note that when we speak about statistical regularities of certain phenomena, we imply more than is usually meant. In particular, we do not mean that these phenomena are described by means of stochastic distributions: as mentioned above, statistical regularities of stochastic phenomena are studied in probability theory, whereas we know almost nothing about statistical regularities of nonstochastic random phenomena.<sup>1</sup>

Let us begin with the simplest case, in which the mass property manifests itself in the form of a sequence. Let  $X$  be an arbitrary set,  $f : X \rightarrow \mathbb{R}$  a real bounded function, and  $\bar{x} \in X^{\mathbb{N}}$  an arbitrary sequence. Suppose further that we are interested in the asymptotic (for  $n \rightarrow \infty$ ) behavior of the averages

$$y_n = \frac{1}{n} \sum_{i=1}^n f(x_i). \quad (4.1)$$

It is clear that they may have no limit. However, by virtue of the boundedness of the function  $f$ , a set of limit points of the sequence  $\{y_n; n \in \mathbb{N}\}$  is automatically not empty. And since for usual probabilistic suppositions this set consists with probability 1 of only one element, the mathematical expectation of the function  $f$ , one can hope that in the general case, the set of limit points will play a role analogous to mathematical expectation. But what in this case will be analogous to the probability distribution?

Associate to each natural number  $n$  and to each subset  $A \subseteq X$  the frequency of hitting in  $A$  in the first  $n$  terms of the sequence  $\bar{x}$ :

$$p_{\bar{x}}^{(n)}(A) = \frac{1}{n} \sum_{i=1}^n \mathbf{1}_A(\bar{x}_i). \quad (4.2)$$

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<sup>1</sup> The author first encountered a random-in-a-broad-sense phenomenon in the far-off year 1960, when he tried to amplify  $10^8$ – $10^9$  times a very weak ( $10^{-8}$ – $10^{-9}$  W) electric signal, a so-called cathode (or shadow) current of a thyratron. According to the physics of the situation, this signal was a weakly correlated random function with normal distribution, almost a white noise, i.e., stochastic, or in other words, a statistically stable phenomenon, generated by nature itself. However, the signal from the output of the amplifier, a device sensitive to everything, even the smallest external perturbations, was already far from being statistically stable or stochastic: its mathematical expectation and dispersion were changing rather significantly. Our intervention by means of the amplifier had destroyed the regularity created by nature. As a result, we had a statistically unstable, nonstochastic phenomenon that happened to require artificial stabilization [33].

We obtain the sequence  $\{p_{\bar{x}}^{(n)}; n \in \mathbb{N}\}$  of probability distributions on  $2^X$  that always has, as will be shown later, a nonempty set of limit points, the finitely additive probability distributions on  $2^X$ . And this set is the sought-for analogue of a probabilistic distribution for the sequence  $\bar{x}$ , which we call the *statistical regularity of the sequence*  $\bar{x}$ . To be more precise, let us denote by  $PF(X)$  the family of all finitely additive probability distributions on the set of all subsets of the set  $X$ , i.e., suppose

$$PF(X) = \{p \in (2^X \rightarrow [0, 1]) : p(X) = 1, p(A \cup B) = p(A) + p(B \setminus A) \\ \forall A, B \subseteq X\}.$$

Further, let  $M(X)$  be a Banach space of real bounded functions on  $X$ ,  $\tau(X)$  a weak-\* topology [13] in  $PF(X)$ , generated by the system of neighborhoods

$$U_{\varepsilon, f_1, f_2, \dots, f_n}(p) = \{p' \in PF(X) : |pf_i - p'f_i| < \varepsilon, \forall i \in \overline{1, n}\},$$

where  $n \in \mathbb{N}$ ,  $\varepsilon > 0$ ,  $f_1, f_2, \dots, f_n \in M(X)$ , and  $pf = \int f(x)p(dx)$  (in the natural sense of the integral over a finitely additive measure).<sup>2</sup>

Denote by  $\mathbf{P}(X)$  the union of all nonempty closed, in the sense of the topology  $\tau(X)$ , subsets of the set  $PF(X)$ , and if  $P \in \mathbf{P}(X)$ ,  $f \in M(X)$ , then we denote by the symbol  $Pf$  the set of numbers  $\{pf; p \in P\}$ .

Associate now to any sequence  $\bar{x} \in X^{\mathbb{N}}$  the set  $P_{\bar{x}}$  of the limit (in the sense of  $\tau(X)$ ) points of the sequence

$$\{p_{\bar{x}}^{(n)}; n \in \mathbb{N}\}.$$

From the compactness of the topological space  $(PF(X), \tau(X))$  it follows that the set  $P_{\bar{x}}$  is not empty, and moreover it is obviously closed. Hence  $P_{\bar{x}} \in \mathbf{P}(X)$ .

We shall call the set  $P_{\bar{x}}$  the *statistical regularity* (or just the *regularity*) of the sequence  $\bar{x}$ . We have established that every sequence has its regularity. It turns out that it suffices to know this regularity in order to “predict” the result of the averaging of an arbitrary function with respect to the considered sequence. Speaking more precisely, the set of limit points of the sequence of numbers  $y_n$  (see (4.1)) can be written in the form a

$$P_{\bar{x}}f = \left\{ \int f(x)p(dx) : \forall p \in P_{\bar{x}} \right\}.$$

Thus, the regularity of an arbitrary sequence plays for this sequence a role similar to that of the probability distribution for a random value. One can speak of a joint regularity of two sequences, etc. It is clear, however, that the set of regularities is “much richer” than the set of sequences. Therefore the following question arises: what is described by the regularities of the general form?

<sup>2</sup> See Appendix A.2.

We call the set

$$X^{(\infty)} = \bigcup_{n=1}^{\infty} X^n, \quad \text{when } X^n = \underbrace{X \times X \times \cdots \times X}_n,$$

the *space of samples from X*, or *sampling space*, and by *sampling directedness in X* we mean any directedness that takes values in the space of samples from X, in other words, the directedness of the form

$$\varphi = \{\varphi_\lambda, \lambda \in \Lambda, \geq\},$$

when

$$\varphi_\lambda = (\varphi_{\lambda_1}, \varphi_{\lambda_2}, \dots, \varphi_{\lambda_{n_\lambda}}) \in X^{n_\lambda}, \quad n_\lambda \in \mathbb{N}.$$

To any sampling directedness  $\varphi$  we will associate the directedness of distributions

$$\{p_\varphi^{(\lambda)}, \lambda \in \Lambda, \geq\},$$

where

$$p_\varphi^{(\lambda)} \in PF(X),$$

$$p_\varphi^{(\lambda)}(A) = \frac{1}{n_\lambda} \sum_{i=1}^{n_\lambda} \mathbf{1}_A(\varphi_{\lambda_i}), \quad A \subseteq X.$$

We call the set of limit points of this directedness of distributions, which obviously is the regularity on X, the *regularity of sampling directedness*  $\varphi$  and denote it by  $P_\varphi$ . It turns out that for a description of any statistical regularity, the concept of sampling directedness is already sufficient. Speaking more precisely, any sampling directedness has a regularity, and any regularity is the regularity of some sampling directedness (Theorem 4.2).

Thus, one can say that the concept of sampling directedness relates to the concept of statistical regularity as in probability theory the concept of random value relates to the concept of probability distribution. And what is more, the set of “average values”  $P_\varphi f = \{pf; p \in P_\varphi\}$  coincides with the set of limit points of the directedness of real numbers

$$y_\lambda = \frac{1}{n_\lambda} \sum_{i=1}^{n_\lambda} f(\varphi_{\lambda_i}), \quad \lambda \in \Lambda.$$

In other words, the mathematical formalism of sampling directedness and its statistical regularity is a generalization of usual notions of random value and its probability distribution on random-in-a-broad-sense phenomena. This formal-

ism includes the usual notions of probability theory as a limiting case (Theorem 4.1).

There is one more road on which the regularities of the general form appear. Denote by  $\mathbf{E}(X)$  the family of all equivalence relations on the set  $X$ , and if  $E \in \mathbf{E}(X)$ , then  $X_{(E)}$  is the set of equivalence classes with respect to  $E$ . Let further

$$\kappa_E : X \rightarrow X_{(E)}, \quad \kappa_E(x) = e \ni x,$$

and if

$$E_1, E_2 \in \mathbf{E}(X), \quad E_1 \subseteq E_2,$$

then

$$\kappa_{E_1 E_2} : X_{(E_1)} \rightarrow X_{(E_2)}, \quad \kappa_{E_1 E_2}(e_1) = e_2 \supseteq e_1.$$

Finally, we shall agree to consider any mapping  $g : Y \rightarrow Z$  predetermined on  $PF(Y)$  in the following way:

$$g(p) = q \in PF(Z), \quad q(A) = p(g^{-1}(A)),$$

for all

$$p \in PF(Y), \quad A \subseteq Z.$$

We shall call any family of regularities of the form

$$\{P_E; E \in \mathcal{E}\},$$

where

$$\mathcal{E} \subseteq \mathbf{E}(X), P_E \in \mathbf{P}(X_{(E)}),$$

and for any

$$E_1, E_2 \in \mathcal{E}$$

there exists

$$E_0 \in \mathcal{E}, \quad E_0 \subseteq E_1 \cap E_2$$

satisfying the condition

$$\kappa_{E_1 E_2}(P_{E_0}) = P_{E_i} \quad (i = 1, 2)$$

a *coordinated family of regularities on  $X$* .

A coordinated family of regularities  $\{P_E; E \subseteq \mathcal{E}\}$  in which all sets  $X_{(E)}$  are finite will be called a *simple coordinated family on  $X$* . It is easy to see that if

$$P \in \mathbf{P}(X), \quad E \in \mathbf{E}(X), \quad P_E = \kappa_E(P),$$

then

$$P_E \in \mathbf{P}(X_E).$$

In this case we shall say that  $\kappa_E(P)$  is a *projection of the regularity  $P$  onto the division  $X_{(E)}$* .

Evidently, the set of all projections of one regularity is a coordinated family of regularities. It turns out that the converse is also true. Any coordinated family of regularities is the family of projections of one or more regularities on  $X$ . We shall call them *preimages* of this coordinated family of regularities. One can show that the union of all preimages of the same coordinated family of regularities is a preimage as well, determined now uniquely. We shall call it a *source*.

Thus we see that any coordinated family of regularities has one and only one source. The converse is trivial: every regularity  $P$  on  $X$  is the source of the coordinated family of regularities consisting of just this regularity.

However, a stronger statement is also true (Theorem 4.3). Any regularity can be presented as a source of some simple coordinated family of regularities, i.e., can be given by means of its projections on finite subdivisions. This is the above-mentioned second way in which the regularities of the general form appear. It is of interest, it seems to us, because the projection of any regularity on a finite subdivision is simply a family of probability distributions on a finite set, while the notion of an arbitrary regularity, even consisting of one distribution on  $2^X$ , is hardly conceivable.

We mention one more consequence of our results. Let

$$X = \prod_{i \in I} X_i,$$

where  $I$  is an arbitrary nonempty set of indices, and all  $X_i$  are arbitrary nonempty sets. Let further

$$P_i \in \mathbf{P}(X_i).$$

Then to every  $i \in I$  one can associate the equivalence relation

$$E_i \in \mathbf{E}(X)$$

given by

$$xE_i y \stackrel{\Delta}{\iff} x_i = y_i.$$

In so doing, to any  $P_i$  there will be naturally associated

$$P_{E_i} \in \mathbf{P}(X_{(E_i)}),$$

and we can expand the notion of coordinated family of regularities on the family of regularities  $P_i$ , defined on different sets  $X_i$ .

Thus it turns out that any coordinated family of regularities, even in such a broad sense, is generated by one and only one source.

## 4.2 The Concept of Statistical Regularity

Let  $X$  be an arbitrary set,  $M(X)$  the Banach space of all real bounded functions on  $X$ ,

$$PF(X) = \{p \in (2^X \rightarrow [0, 1]) : p(X) = 1, p(A \cup B) = p(A) + p(B \setminus A) \\ \forall A, B \subseteq X\}.$$

Let us consider  $PF(X)$  as a subset of an adjoint space  $(M(X))^*$  and define on  $PF(X)$  the topology  $\tau(X)$  that is the trace of the weak-\* topology in  $(M(X))^*$ . In other words, denoting, for brevity, the integral of the function  $f \in M(X)$  over the finitely additive measure  $p \in PF(X)$  by  $pf$ , we declare as a fundamental system of neighborhoods in  $PF(X)$  the sets

$$U_{\varepsilon, f_1, f_2, \dots, f_n}(p) = \{p' \in PF(X) : |pf_i - p'f_i| < \varepsilon, \forall i \in \overline{1, n}\},$$

where  $n \in \mathbb{N}$ ,  $\varepsilon > 0$ ,  $f_1, f_2, \dots, f_n \in M(X)$ .

Finally, for an arbitrary  $P \subseteq PF(X)$ ,  $f \in M(X)$ , assume that

$$Pf = \{r \in \mathbb{R} : \exists p \in P, pf = r\}.$$

**Definition 4.1.** Let  $X$  be an arbitrary set. We call any nonempty closed set  $P$  of the topological space  $(PF(X), \tau(X))$  a *statistical regularity* on  $X$ . We denote the family of all regularities on  $X$  by  $\mathbf{P}(X)$ .

If  $P \in \mathbf{P}(X)$ ,  $F \subseteq M(X)$ , we say that the regularity  $P$  is  $\mu$ - $F$ -stochastic (or simply  $F$ -stochastic) if there are a  $\sigma$ -algebra  $\mathcal{A}$  of subsets of the set  $X$  and a probability (countable additive) measure  $\mu$  on  $\mathcal{A}$  such that

$$pf = \int f(x)\mu(dx)$$

for all  $p \in P$ ,  $f \in F$ .

As is known (see, for example, [13]), the topological space  $(PF(X), \tau(X))$  is compact for any  $X$ . Therefore, the sequence

$$\{p_x^{(n)}; n \in \mathbf{N}\},$$

constructed in accordance with the formula (4.2), always has a nonempty set of limit points  $P_x \in \mathbf{P}(X)$ , which we call the *regularity of the sequence*  $x \in X^{\mathbf{N}}$ . The connection between these notions and usual notions of probability theory is established by the following theorem.

**Theorem 4.1.** *Let  $(X, \mathcal{A}, \mu)$  be a probability space,  $\xi = \{\xi_n, n \in \mathbf{N}\}$  a sequence of independent random elements taking values in  $X$  with the probability distribution  $\mu$ , and  $F \subseteq M(X)$  a finite or countable family of  $\mu$ -measurable functions. Then the regularity of the sequence of values  $\xi$  will be  $\mu$ - $F$ -stochastic with probability 1.*

*Proof.* Obviously, by virtue of the boundedness of the functions  $f \in F$  and countability of  $F$ , there is a countable system  $\mathcal{B}$  of subsets of the set  $X$  such that all  $B \in \mathcal{B}$  are  $\mu$ -measurable sets, and for any  $\varepsilon > 0$ ,  $f \in F$ , one can produce a finite subdivision  $E$  of the set  $X$  into nonoverlapping subsets satisfying the conditions  $E \subseteq \mathcal{B}$ ,  $\forall e \in E$ ,

$$\sup_{x \in e} f(x) - \inf_{x \in e} f(x) < \varepsilon.$$

Denote by  $(\Omega, \Sigma, \pi)$  the probability space on which all  $\xi_n$  are determined, and further, set

$$\Omega_A = \left\{ \omega \in \Omega : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \mathbf{1}_A(\xi_i(\omega)) = \mu(A) \right\}, \quad A \in \mathcal{A}.$$

For any  $B \in \mathcal{B}$ ,  $m \in \mathbf{N}$ , one can find measurable sets  $\underline{A}_{B,m}$  and  $\bar{A}_{B,m}$  such that

$$\underline{A}_{B,m} \subseteq B \subseteq \bar{A}_{B,m}$$

and

$$\mu(\bar{A}_{B,m} \setminus \underline{A}_{B,m}) < \varepsilon.$$

Let

$$\Omega_0 = \bigcap_{B \in \mathcal{B}} \bigcap_{m=1}^{\infty} \left( \Omega_{\underline{A}_{B,m}} \cap \Omega_{\bar{A}_{B,m}} \right).$$

Then from  $\omega \in \Omega_0$  it follows that

$$P_{\xi(\omega)}(B) = \mu(B), \quad \forall B \in \mathcal{B},$$

and thus

$$P_{\xi(\omega)}f = \left\{ \int_X f(x)\mu(dx) \right\} \quad \forall f \in F.$$

It remains only to obtain, making use of the strong law of large numbers, that  $\pi(\Omega_0) = 1$ . The theorem is proved.

It is clear, however, that only the regularity concentrated on a countable subset of the set  $X$  can be the regularity of some sequence. Indeed, let  $\bar{x} \in X^{\mathbb{N}}$ ,  $P_{\bar{x}} = P$ . Denote by  $X_0 = \bar{x}(\mathbb{N})$  the set of values of the sequence  $\bar{x}$ . This set is at most countable. Let us show that for any  $p \in P$  the relation  $p(X_0) = 1$  is correct.

By the definition of the set  $P$ , for any  $\varepsilon > 0$ ,  $n_0 \in \mathbb{N}$ ,  $f \in M(X)$ , there is  $n \geq n_0$  such that

$$\left| p_{\bar{x}}^{(n)}f - pf \right| < \varepsilon.$$

In particular, if  $f = \mathbf{1}_{X \setminus X_0}$ , i.e.,

$$f(x) = \begin{cases} 0, & \text{for } x \in X_0, \\ 1, & \text{for } x \in X \setminus X_0, \end{cases}$$

we obtain

$$\left| p_{\bar{x}}^{(n)}(X \setminus X_0) - p(X \setminus X_0) \right| < \varepsilon.$$

But

$$p_{\bar{x}}^{(n)}(X \setminus X_0) = 0,$$

and hence for any  $p \in P$ ,  $\varepsilon > 0$ , the inequality  $p(X \setminus X_0) < \varepsilon$  holds. Hence  $p(X_0) = 1$ .

Therefore the question arises whether there are more general constructions resulting in regularities of general form. Here we investigate two such constructions. The third construction is considered in Chapter 5.

### 4.3 Sampling Directedness and Statistical Regularity

Let us begin by recalling the definition of directedness. A direction on the set  $\Lambda$  is a binary relation  $\geq$  on  $\Lambda$  possessing the following properties:

1. If  $\lambda_1, \lambda_2, \lambda_3 \in \Lambda$ ,  $\lambda_1 \geq \lambda_2$  and  $\lambda_2 \geq \lambda_3$ , then  $\lambda_1 \geq \lambda_3$  (transitivity).
2. If  $\lambda \in \Lambda$ , then  $\lambda \geq \lambda$  (reflectivity).
3. If  $\lambda_1, \lambda_2 \in \Lambda$ , then there is an element  $\lambda_3 \in \Lambda$  such that  $\lambda_3 \geq \lambda_1$  and  $\lambda_3 \geq \lambda_2$ .

A *directed set* is an ordered pair  $(\Lambda, \geq)$  consisting of an arbitrary set  $\Lambda$  and a direction  $\geq$  on it. A *directedness* is an ordered pair  $(\varphi, \geq)$ , where  $\varphi$  is some function, and  $\geq$  is the direction on the domain of definition of this function. If  $(\varphi, \geq)$  is the directedness,  $\varphi : \Lambda \rightarrow Y$ , and  $\tau$  is the fixed topology in  $Y$ , then we say that the point  $y \in Y$  is the limit of the directedness  $(\varphi, \geq)$  if for any neighborhood of that point one can find  $\lambda_0 \in \Lambda$  such that  $\varphi(\lambda)$  is situated in this neighborhood for all  $\lambda \geq \lambda_0$ . It is written in the form  $y = \lim(\varphi, \geq)$ . We call the point  $y \in Y$  the limit point of the directedness if for any  $\lambda_0 \in \Lambda$  and any neighborhood of the point one can find  $\lambda \geq \lambda_0$  such that  $\varphi(\lambda)$  is contained in this neighborhood. We denote the set of all limit points (perhaps empty) by  $\text{LIM}(\varphi, \geq)$ . Finally, in those cases in which we have to indicate explicitly the domain of definition  $\Lambda$  of the directedness  $(\varphi, \geq)$ , we shall use for this purpose the notation  $(\varphi|\Lambda, \geq)$  or  $(\varphi_\lambda, \lambda \in \Lambda, \geq)$ .

Let, as earlier,  $X$  be an arbitrary set. By  $X^\infty$  we denote the “space of samples from  $X^n$ ”:

$$X^{(\infty)} = \bigcup_{n=1}^{\infty} X^n,$$

where

$$X^n = \underbrace{X \times X \times \cdots \times X}_n.$$

**Definition 4.2.** We call any directedness  $(\varphi, \geq)$  taking values in the space of samples from  $X$  a *sampling directedness* in  $X$ .

To each sampling directedness  $(\varphi|\Lambda, \geq)$  we associate the directedness of natural numbers  $(n_\varphi|\mathbb{N}, \geq)$  and the directedness of distributions  $(p_\varphi|\mathcal{P}, \geq)$ . The last two are defined in the following way:  $n_\varphi(\lambda)$  is  $n \in \mathbb{N}$  such that  $\varphi(\lambda) \in X^n$ , and  $p_\varphi(\lambda)$  is the distribution from  $PF(X)$  such that

$$(p_\varphi(\lambda))(A) = \frac{1}{n_\varphi(\lambda)} \sum_{i=1}^{n_\varphi(\lambda)} \mathbf{1}_A(\varphi_i(\lambda)),$$

where

$$A \subseteq X, \quad (\varphi_1, \varphi_2, \dots, \varphi_{n_\varphi(\lambda)}) = \varphi(\lambda).$$

We call the set  $P_\varphi$  of limit points of the directedness  $(p_\varphi, \geq)$ , i.e.,

$$P_\varphi = \text{LIM}(p_\varphi, \geq),$$

the *statistical regularity* (SR) of the sampling directedness.

It is obvious now that to any sequence  $\bar{x} \in X^{\mathbb{N}}$  one can assign a sampling directedness  $(\varphi \mid \mathbb{N}, \geq)$  in  $X$ , where  $\geq$  is the natural direction in  $\mathbb{N}$  and  $\varphi(n) = (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n)$ . Here  $p_\varphi(n) = p_{\bar{x}}(n)$  ( $n \in \mathbb{N}$ ), and so  $P_\varphi = P_{\bar{x}}$ . It turns out that the notion of sampling directedness generalizes the notion of an arbitrary sequence exactly as we need.

**Theorem 4.2.** *Any sampling directedness has a regularity, and any regularity is the regularity of some sampling directedness. Furthermore, the set*

$$P_\varphi f = \{pf; p \in P_\varphi\},$$

where  $f \in M(X)$  and  $\varphi$  is the sampling directedness in  $X$ , coincides with the set of limit points of the directedness of average values

$$y_\lambda = \frac{1}{n_\lambda} \sum_{i=1}^{n_\lambda} f(\varphi_{\lambda_i}), \quad \lambda \in \Lambda,$$

i.e.,

$$P_\varphi f = \text{LIM}\{y_\lambda, \lambda \in \Lambda, \geq\}.$$

*Remark 4.1.* Finitely additive measures have been attracting ever greater attention from many researchers (see [9, 86, 88], for example). In this connection it seems interesting that in the theorem formulated above and also in Theorem 5.2 from the next chapter, finitely additive probabilities appear not as an independent research object but “out of necessity” when one is considering such natural notions as sequence, directedness, and the criterion choice rule [37, 38, 24].

*Proof (of Theorem 4.2).* If  $(\varphi, \geq)$  is the sampling directedness in  $X$ , then  $P_\varphi \neq \emptyset$  follows immediately from the compactness of the topological space  $(PF(X), \tau(X))$ . That the set  $P_\varphi$  is closed follows from the fact that it is the set of limit points of the directedness  $(p_\varphi, \geq)$ . Indeed, if  $p \in PF(X) \setminus P_\varphi$ , then for some  $\varepsilon > 0$  one can find  $\lambda_0 \in \Lambda$  and a finite set of functions  $f_1, f_2, \dots, f_k \in M(X)$  such that for all  $\lambda \geq \lambda_0$  some  $i \in \overline{1, k}$  exists such that  $|pf_i - p_\varphi(\lambda)f_i| > 2\varepsilon$ . But in this case, for any  $p_0 \in U_{\varepsilon, f_1, f_2, \dots, f_k}(p)$  we have

$$2\varepsilon < |pf_i - p_\varphi(\lambda)f_i| \leq |pf_i - p_0f_i| + |p_0f_i - p_\varphi(\lambda)f_i| < \varepsilon + |p_0f_i - p_\varphi(\lambda)f_i|,$$

and thus

$$|p_0f_i - p_\varphi(\lambda)f_i| > \varepsilon,$$

i.e.,

$$p_0 \in PF(X) \setminus P_\varphi.$$

Now let  $P$  be a nonempty closed set in  $PF(X)$ . Our purpose is to construct a sampling directedness  $(\varphi, \geq)$  such that  $P_\varphi = P$ . For that denote by  $Q$  the set of all distributions  $q \in PF(X)$ , satisfying two conditions:

- (i)  $q$  is concentrated on some finite subset  $X_q$  of the set  $X$ ;
- (ii) all numbers  $q(x)$  ( $x \in X$ ) are rational.

It is easily checked that to every  $q \in Q$  one can associate  $n_q \in \mathbb{N}$  and  $x_1^q, x_2^q, \dots, x_{n_q}^q \in X_q$  in such a way that

$$q(A) = \frac{1}{n_q} \sum_{i=1}^{n_q} \mathbf{1}_A(x_i^q) \quad (A \subseteq X).$$

Indeed, assuming without the loss of generality that  $q(x) > 0$  for all  $(x \in X_q)$ , we present all  $q(x)$  ( $x \in X_q$ ) in the form of irreducible fractions and denote by  $n_q$  the least common multiple of their denominators. Reduce all these fractions to the common denominator  $n_q$ . Define

$$q(x) = m_q(x)/n_q.$$

And now, since

$$\sum_{x \in X_q} m_q(x) = n_q,$$

we can take a collection of elements  $\{x_i^q, i \in \overline{1, n_q}\}$  in which every  $x \in X_q$  occurs exactly  $m_q(x)$  times.

We can now show that the set  $Q$  is everywhere dense in the space  $(PF(X), \tau(X))$ . For that purpose, taking arbitrary  $p \in PF(X)$  and  $f_1, f_2, \dots, f_n \in M(X)$ , note that for any  $\varepsilon > 0$  one can find a subdivision  $E$  of the set  $X$  into a finite number of disjoint nonempty subsets

$$(E = \{e_1, e_2, \dots, e_s\})$$

such that for any  $e \in E$  and any  $i \in \overline{1, n}$  the inequality

$$\sup f_i(e) - \inf f_i(e) < \varepsilon$$

holds.

Now in every set  $e \in E$  choose one arbitrary point  $x_e$  and set

$$k = \max \left\{ 1, \max_{1 \leq i \leq n} \|f_i\| \right\},$$

$$E_0 = \{e \in E : \text{the number } p(e) \text{ is irrational}\}.$$

If  $E_0 = \emptyset$ , then assuming  $q(x_e) = p(e), \forall e \in E$ , we obtain

$$q \in \mathcal{Q}, \quad |qf_i - pf_i| < \varepsilon \quad \forall i \in \overline{1, n}.$$

If  $E_0 \neq \emptyset$ , then we set

$$\delta_1 = \min_{e \in E_0} p(e), \quad \delta_2 = \min_{e \in E_0} (1 - p(e)), \quad \delta = \min\{\delta_1, \delta_2\}.$$

Obviously  $0 < \delta < 1$ . Therefore for any  $e \in E_0$  one can find a rational number  $r_e$  such that

$$p(e) - \frac{\delta}{[\text{Card}(E)]^2 k} < r_e < p(e),$$

and then, fixing an arbitrary set  $e_0 \in E_0$ , assume that

$$q(x_e) = \begin{cases} p(e), & \text{if } e \in E \setminus E_0, \\ r_e, & \text{if } e \in E_0 \setminus \{e_0\}, \\ \sum_{e \in E_0} p(e) - \sum_{e \neq e_0} r_e, & \text{if } e = e_0. \end{cases}$$

We obtain

$$q \in \mathcal{Q}, \quad |q(e) - p(e)| < \frac{\delta}{\text{Card}(E)k} \quad \forall e \in E$$

and thus

$$|qf_i - pf_i| \leq \frac{\delta \|f_i\|}{k} + \varepsilon \leq \delta + \varepsilon \leq 2\varepsilon.$$

By virtue of the arbitrariness of  $\varepsilon > 0$ , this proves that the set  $\mathcal{Q}$  is dense in the space  $(PF(X), \tau(X))$ .

Now let  $P \in \mathbf{P}(X)$  be an arbitrary regularity on  $X$ . Define

$$\mathbb{R}^+ = ]0, \infty[, \quad M^{(\infty)} = \bigcup_{n=1}^{\infty} (M(X))^n,$$

where

$$(M(X))^n = \underbrace{M(X) \times M(X) \times \cdots \times M(X)}_n,$$

and

$$\Lambda = \mathbb{R}^+ \times M^{(\infty)} \times P.$$

On the set  $\Lambda$  introduce the direction  $\geq$  by the formula

$$\begin{aligned}
(\varepsilon_1, f_{11}, f_{12}, \dots, f_{1n_1}, p_1) &\geq (\varepsilon_2, f_{21}, f_{22}, \dots, f_{2n_2}, p_2) \\
&\Leftrightarrow (\varepsilon_1 \leq \varepsilon_2, \{f_{11}, f_{12}, \dots, f_{1n_1}\} \supseteq \{f_{21}, f_{22}, \dots, f_{2n_2}\})
\end{aligned}$$

(in this case no condition is imposed on  $p_1$  and  $p_2$ ).

Associate now to any

$$\lambda = (\varepsilon, f_1, f_2, \dots, f_n, p) \in \Lambda$$

some distribution

$$q_\lambda \in \mathcal{Q} \cap U_{\varepsilon, f_1, f_2, \dots, f_n}(p),$$

which exists by virtue of the denseness of the set  $\mathcal{Q}$ . Some collection of points

$$\{x_1^q, x_2^q, \dots, x_{n_q}^q\} \in X^{n_q}$$

corresponds, as was shown above, to this  $q = q_\lambda$ .

We denote this collection by  $\varphi(\lambda)$  and thus arrive at the sampling directedness  $\{\varphi, \geq\}$ , where

$$\varphi : \Lambda \rightarrow X^{(\infty)}, \quad \lambda \mapsto \varphi(\lambda).$$

It remains to prove that  $P_\varphi = P$ .

First, let the point  $p$  belong to the set  $P$ , and we have to prove that  $p \in P_\varphi$ , i.e., that for any  $\varepsilon > 0$ , any collection of functions

$$f_1, f_2, \dots, f_n \in M(X),$$

and the point

$$\lambda_0 = (\varepsilon_1, f_{01}, f_{02}, \dots, f_{0n_0}, p_0)$$

one can find a point

$$\lambda_1 = (\varepsilon_1, f_{11}, f_{12}, \dots, f_{1n_1}, p_1)$$

such that

$$\lambda_1 \geq \lambda_0$$

and

$$p_\varphi(\lambda_1) \in U_{\varepsilon, f_1, f_2, \dots, f_n}(p).$$

But this is obvious: it suffices to take

$$\begin{aligned}\varepsilon_1 &= \min\{\varepsilon, \varepsilon_0\}, \\ \{f_{11}, f_{12}, \dots, f_{1n_1}\} &= \{f_1, f_2, \dots, f_n, f_{01}, f_{02}, \dots, f_{0n_0}\}, \\ p_1 &= p_2,\end{aligned}$$

and we obtain by definition

$$\lambda_1 \geq \lambda_0, \quad p_\varphi(\lambda_1) = q_{\lambda_1} \in U_{\varepsilon_1, f_{11}, f_{12}, \dots, f_{1n_1}}(p) \subseteq U_{\varepsilon, f_1, f_2, \dots, f_n}(p).$$

Now suppose that  $p \in P_\varphi \setminus P$ . In this case, since the set  $P$  is closed, there exist  $\varepsilon > 0$ ,  $n \in \mathbb{N}$ ,  $f_1, f_2, \dots, f_n \in M(X)$ , such that for any  $p_1 \in P$ , an index  $i \in \overline{1, n}$  can be found for which

$$|p_1 f_i - p f_i| > \varepsilon. \quad (4.3)$$

On the other hand, since

$$p \in P_\varphi = \text{LIM}(p_\varphi, \geq),$$

for any  $\lambda \in \varepsilon$  one can find  $\lambda > \lambda_0$  such that

$$|p_\varphi(\lambda_i) f_i - p f_i| < \varepsilon/2 \quad \forall i \in \overline{1, n}. \quad (4.4)$$

Let us take a concrete

$$\lambda_0 = (\varepsilon/2, f_1, f_2, \dots, f_n, p_0)$$

and write in detail what in this case would be  $\lambda_1$  appearing in (4.4):

$$\lambda_1 = (\varepsilon_1, f_{11}, f_{12}, \dots, f_{1n_1}, p_1),$$

where

$$\varepsilon_1 \leq \varepsilon/2, \quad f_{11}, f_{12}, \dots, f_{1n_1} \supseteq f_1, f_2, \dots, f_n.$$

However, by the construction of the sampling directedness  $(\varphi, \geq)$  we have

$$|p_\varphi(\lambda_1) f_{1j} - p_1 f_{1j}| < \varepsilon_1 \quad \forall j \in \overline{1, n_1},$$

and hence

$$|p_\varphi(\lambda_1) f_i - p_1 f_i| < \varepsilon/2 \quad \forall i \in \overline{1, n}.$$

But this, in combination with relation (4.4), contradicts relation (4.3). This proves that  $P_\varphi = P$ .

To complete the proof of the theorem it remains to demonstrate that

$$P_\varphi f = \text{LIM} \{y_\lambda, \lambda \in \Lambda, \geq\} = \text{LIM}(y, \geq).$$

By definition,

$$(p_\varphi(\lambda))f = y_\lambda,$$

so that if  $a \in \text{LIM}(y, \geq)$ , then

$$a = \lim(y|\tilde{\Lambda}, \geq) = \lim((p_\varphi(\cdot))f|\tilde{\Lambda}, \geq),$$

for some cofinal subset  $\tilde{\Lambda} \subset \Lambda$ , and thus

$$a \in \text{LIM}(p_\varphi(\cdot)f|\Lambda, \geq),$$

i.e.,

$$P_\varphi f \supseteq \text{LIM}(y, \geq).$$

The inverse inclusion is proved similarly. The theorem is proved.

#### 4.4 Coordinated Families of Regularities

In order to describe the second construction leading to the regularities of general form, we shall agree to associate with any subdivision  $E$  of the set  $X$  into disjoint subsets the equivalence relation  $\tilde{E}$ , generated by this subdivision, i.e.,

$$\tilde{E} = \{(x_1, x_2) \in X^2 : \exists e \in E, x_1, x_2 \in e\}.$$

Denote the family of all equivalence relations on  $X$  by  $\tilde{\mathbf{E}}(\mathbf{X})$ , and denote by  $\tilde{E}^*(X)$  the subfamily of  $\tilde{E} \in \tilde{\mathbf{E}}(X)$  for which the set  $E$  of equivalence classes is finite. Analogously, the family of all subdivisions and the family of finite subdivisions will be denoted by  $\mathbf{E}(X)$  and  $\mathbf{E}^*(X)$ .

Furthermore, when

$$\tilde{E}, \tilde{E}_1, \tilde{E}_2 \in \tilde{\mathbf{E}}(X), \quad \tilde{E}_1 \subseteq \tilde{E}_2,$$

we shall consider the canonical mappings

$$\kappa_E : X \rightarrow E, \quad \kappa_E(x) = e \in E,$$

where  $e$  is determined by

$$x \in e, \quad \kappa_{E_1 E_2} : E_1 \rightarrow E_2, \quad \kappa_{E_1 E_2}(e_1) = e_2, \quad e_1 \subseteq e_2.$$

Each mapping  $g : Y \rightarrow Z$  is predetermined on  $PF(Y)$  in the following way:

$$g(p) = q \in PF(Z)$$

such that

$$q(A) = p(g^{-1}(A))$$

for any

$$p \in PF(Y), \quad A \subseteq Z.$$

In other words,  $g(p)$  is the image of the measure  $p$  for the mapping  $g$  in the usual sense: if  $P \subseteq PF(Y)$ ,  $g : Y \rightarrow Z$ , then  $g(P) = \{g(p); p \in P\}$ .

**Definition 4.3.** A *coordinated family of regularities on  $X$*  is any family of regularities of the form

$$\{P_E; E \in \mathcal{E}\},$$

where

$$\mathcal{E} \subseteq \mathbf{E}(X), \quad P_E \in \mathbf{P}(E),$$

satisfying the following condition: for any  $E_1, E_2 \in \mathcal{E}$  there exists  $E_0 \in \mathcal{E}$  such that

$$\tilde{E}_0 \subseteq \tilde{E}_1 \cap \tilde{E}_2,$$

and at the same time,

$$\kappa_{E_0 E_i}(P_{E_0}) = P_{E_i} \quad (i = 1, 2).$$

We shall call a coordinated family of regularities  $\{P_E; E \in \mathcal{E}\}$  in which all sets  $E$  are finite a *simple coordinated family* on  $X$ . We shall say that  $P$  is the *source* of the coordinated family of regularities  $\{P_E; E \in \mathcal{E}\}$  if the following conditions are satisfied:

- (1)  $P \in \mathbf{P}(X)$ ;
- (2)  $\kappa_E(P) = P_E$  for all  $E \in \mathcal{E}$ ;
- (3) if  $p \in PF(X)$  and  $\kappa_E(p) \in P_E$  for all  $E \in \mathcal{E}$ , then  $p \in P$ .

**Theorem 4.3.** *Every coordinated family of regularities has one and only one source, and every regularity on  $X$  is a source of some simple coordinated family.*

*Proof.* For arbitrary

$$E \in \mathbf{E}(X), \quad P_E \in \mathbf{P}(E)$$

the set  $\kappa_E^{-1}(P_E)$ , which for brevity we shall denote by  $P_E^{-1}$ , is a regularity on  $X$ .

Let us prove this. Let  $g \in M(E)$ . Then, obviously, the function  $f = g \circ \kappa_E$  belongs to  $M(X)$ , so that

$$M = \{f \in M(X) : \exists g \in M(E), f = g \circ \kappa_E\}$$

is a linear subset of the set  $M(X)$  and to every function  $f \in M(E)$  one can associate one and only one function  $g_f \in M(E)$  such that  $g_f \circ \kappa_E = f$ , and for any

$$p \in PF(X), \quad f \in M(E),$$

the relation

$$(\kappa_E^{-1}(p))f = pg_f$$

is true.

Associate to an arbitrary  $p \in P_E$  the linear functional

$$l_p : M(E) \rightarrow \mathbb{R}, \quad l_p(f) = pg_f.$$

Since

$$\mathbf{1}_X \in M(E), \quad l_p(\mathbf{1}_E) = 1,$$

and since  $l_p(f) \geq 0$  when  $f \geq 0$ , it follows that  $\|l_p\| = 1$ . Continue  $l_p$  with conservation of the norm on all  $M(X)$  and denote this continuation by  $l$ . We have to make sure that  $l(f) \geq 0$  for any nonnegative function  $f \in M(X)$ .

Suppose the contrary, i.e., that there is a function  $f \geq 0$  such that

$$l(f) < 0.$$

Without loss of generality one can suppose that

$$\sup_{x \in X} f(x) = 1.$$

In this case we have

$$0 < \mathbf{1}_X(x) - f(x) \leq 1, \quad l(\mathbf{1}_X - f) = 1 - l(f) > 1,$$

and this means that

$$\|l\| = \sup_{|f_1| < 1} \frac{l(f_1)}{\|f_1\|} \geq \frac{l(\mathbf{1}_X - f)}{\|\mathbf{1}_X - f\|} > 1,$$

in spite of the condition

$$\|l\| = \|l_p\| = 1.$$

It remains to assume that

$$p_0(A) = l(\mathbf{1}_A) \quad \forall A \subseteq X$$

and to make sure that

$$p_0 \in PF(X) \quad \text{and} \quad \kappa_E(p_0) \in P_E,$$

since if

$$B \in E,$$

then

$$p_0 = (\kappa_E^{-1}(B)) = l(\mathbf{1}_{\kappa_E^{-1}(B)}) = l_p(\mathbf{1}_{\kappa_E^{-1}(B)}) = p(B).$$

So  $p_0 \in P_E^{-1} \neq \emptyset$ .

Completeness of the set  $P_E^{-1}$  follows from the fact that if

$$p_1 \in PF(X) \setminus P_E^{-1},$$

then

$$p_2 = \kappa_E(p_1) \in PF(E) \setminus P_E,$$

and therefore one can find

$$\varepsilon > 0, \quad g_1, g_2, \dots, g_n \in M(E)$$

such that for any  $p \in P_E$  one can always find  $i \in \overline{1, n}$  such that

$$|pg_i - p_2g_i| > \varepsilon.$$

Setting

$$f_i = g_i \circ \kappa_E \quad (i \in \overline{1, n}),$$

we obtain that

$$|\kappa_E^{-1}(p)f_i - p_1f_i| > \varepsilon \quad (i \in \overline{1, n}).$$

But this means that in  $PF(X)$  there is a neighborhood of the point  $p_1$  that does not intersect the set  $P_E^{-1}$ . Thus we have proved that

$$P_E^{-1} \in \mathbf{P}(X).$$

Now the following lemma will be necessary.

**Lemma 4.1.** *Let*

$$\{P_E; E \in \mathcal{E}\}$$

be the coordinated family of regularities

$$n \in \mathbb{N}, \quad E_1, E_2, \dots, E_n \in \mathcal{E}.$$

Then

$$\bigcap_{i=1}^n P_{E_i}^{-1} \in \mathbf{P}(X).$$

*Proof.* It follows from the condition of Theorem 4.3 that there exists a subdivision

$$E_0 \in \mathcal{E}$$

such that

$$\tilde{E}_0 \subseteq \bigcap_{i=1}^n E_i,$$

and at the same time,

$$P_{E_i} = \kappa_{E_0 E_j}(P_{E_0}) \quad \forall i \in \overline{1, n}.$$

Indeed, if  $n = 2$ , this statement is contained in the definition itself. If one assumes that it is true for some fixed collection  $E'_1, E'_2, \dots, E'_n$ , then applying it to an arbitrary  $E_{n+1} \in \mathcal{E}$  and  $E'_0$  constructed on  $E'_1, E'_2, \dots, E'_n$ , we obtain that the desired  $E_0$  does indeed exist.

Note that nonemptiness of the set  $P_{E_0}^{-1}$ , proved above, immediately implies that

$$\kappa_{E_i}(P_{E_0}^{-1}) = \kappa_{E_0 E_i}(\kappa_{E_0}(P_{E_0}^{-1})) = P_{E_i} \text{ for all } i \in \overline{1, n},$$

i.e.,

$$P_{E_0}^{-1} \subseteq \bigcap_{i=1}^n P_{E_i}^{-1} \neq \emptyset.$$

Completeness of the set

$$\bigcap_{i=1}^n P_{E_i}^{-1}$$

follows from the fact that if

$$p \in PF(X) \setminus \bigcap_{i=1}^n P_{E_i}^{-1},$$

then  $i \in \overline{1, n}$  can be found such that

$$p \in PF(X) \setminus P_{E_i}^{-1}.$$

Thus, as was proved above, there is a neighborhood  $U_i$  of the point  $p$  that does not intersect  $P_{E_i}^{-1}$ . Then, a fortiori, it does not intersect

$$\bigcap_{i=1}^n P_{E_i}^{-1},$$

and hence the statement of the lemma is true.

Returning to the proof of Theorem 4.3, note that according to Lemma 4.1,  $\{P_E^{-1}; E \in \mathcal{E}\}$  is the centered family of closed sets in the compact topological space  $(PF(X), \tau(X))$ . Hence it has a nonempty intersection

$$P = \bigcap_{E \in \mathcal{E}} P_E^{-1},$$

which by virtue of its obvious completeness is a regularity on  $X$ . It remains to note that if

$$p \in PF(X) \setminus P,$$

then one can find

$$E \in \mathcal{E}$$

such that

$$p \notin P_E^{-1},$$

and hence

$$\kappa_E(p) \notin P_E.$$

The first statement of Theorem 4.3 is proved.

In order to prove the second statement, choose as  $\mathcal{E}$  the family of all subdivisions  $E$  having a finite number of elements. Let  $P$  be an arbitrary regularity on  $X$ ,

$$P_E = \kappa_E(P) \quad \forall E \in \mathcal{E}.$$

Obviously,

$$\{P_E, E \in \mathcal{E}\}$$

is a coordinated family of regularities, and we have only to prove that  $P$  is its source. To put it more precisely, the only fact that needs to be proved is that if

$$p \in PF(X) \setminus P,$$

then one can find  $E \in \mathcal{E}$  such that

$$\kappa_E(P) \notin P_E.$$

Fix  $\varepsilon > 0$  and functions

$$f_1, f_2, \dots, f_n \in M(X)$$

such that the neighborhood  $U_{\varepsilon, f_1, f_2, \dots, f_n}(p)$  does not intersect  $P$ .

Let

$$S = \sup \{|f_i(x)|; i \in \overline{1, n}, x \in X\}, \quad K = \left[ \frac{3S}{\varepsilon} \right] + 1,$$

where the brackets denote the integral part of a number, and

$$\begin{aligned} \tilde{E} = \left\{ (x_1, x_2) \in X^2 : \forall i \in \overline{1, n}, \exists k \in \overline{1, K}, \forall j \in (1, 2), \right. \\ \left. -S + (k-1)\frac{\varepsilon}{3} \leq f_i(x_j) < -S + k\frac{\varepsilon}{3} \right\}, \end{aligned}$$

i.e., the subdivision  $E$  is such that for any  $i \in \overline{1, n}$  the diameter  $f_i$  of the image of any element  $e \in E$  does not exceed  $\frac{\varepsilon}{3}$ . Finally, define

$$g_i : E \rightarrow R, \quad g_i(e) = \sup_{x \in e} f_i(x), \quad i \in \overline{1, n}.$$

Then we have

$$|p' f_i - \kappa_E(p') g_i| < \frac{\varepsilon}{3}$$

for any

$$p' \in PF(X), \quad i \in \overline{1, n},$$

and so if  $p_1 \in P$ , then one can find  $i \in \overline{1, n}$  such that

$$|\kappa_E(p_1) g_i - \kappa_E(p) g_i| > \frac{\varepsilon}{3}.$$

Thus

$$\kappa_E(p) \notin \kappa_E(P) = P_E,$$

and the theorem is completely proved.

## 4.5 Statistical Regularity of a Sequence

The structure of statistical regularity of a statistically unstable sequence with values from an arbitrary set is quite complicated, because the corresponding sequence of frequencies may have subdirectednesses that are not subsequences of the original sequence. But in the case of a sequence with values from a finite set, the situation is simplified [91].

Let  $\Omega$  be a finite set, i.e.,  $\Omega = \{\omega_1, \omega_2, \dots, \omega_n\}$ ,  $\bar{\Theta}$  is some sequence of elements from  $\Omega$ ,  $P(\bar{\Theta})$  the statistical regularity of the sequence  $\bar{\Theta}$ .

**Theorem 4.4.** *The measure  $p \in PF(\Omega)$  belongs to the statistical regularity  $P(\bar{\Theta})$  if and only if in the sequence of measures  $\{p_n\}$ ,  $p_n(A) = \sum_{i=1}^n \mathbf{1}_A(\bar{\theta}_i)$ , there exists a subsequence  $\{p_{n_k}\}$  such that for any  $A \in 2^\Omega$ ,*

$$\lim_{k \rightarrow \infty} p_{n_k}(A) = P(A).$$

*Proof.* Necessity: Let  $p \in P(\bar{\Theta})$  and consider a neighborhood  $v^k(p)$  of the point  $p$  of the topological space  $(PF(\Omega), \tau(\Omega))$ :

$$v^k(p) = \left\{ p' \in PF(\Omega) : |pf - p'| < \frac{1}{k} \quad \forall f \in \{I_A : A \in 2^\Omega\} \right\},$$

where  $I_A$  is the indicator of the set  $A$ .

Since  $p$  is the limit point of the sequence of measures  $P_n$ , one can choose a subsequence  $N_k$  of the sequence of natural numbers such that  $p_{n_k} \in v^k(p)$  at any  $k \in \mathbb{N}$ . Therefore

$$\lim_{k \rightarrow \infty} p_{n_k}(A) = p(A) \quad \text{at any } A \in 2^\Omega.$$

Sufficiency: Let

$$\lim_{k \rightarrow \infty} p_{n_k}(A) = p(A) \quad \forall A \in 2^\Omega.$$

We must show that  $p \in P(\bar{\Theta})$ .

Any bounded function  $f : \Omega \rightarrow \mathbb{R}$  may be represented in the form

$$f = \lambda_1 I_{\omega_1} + \lambda_2 I_{\omega_2} + \dots + \lambda_m I_{\omega_m} \quad \text{where } \lambda_i = f(\omega_i),$$

and

$$I_{\omega_i}(\xi) = \begin{cases} 1, & \text{if } \xi = \omega_i, \\ 0, & \text{otherwise,} \end{cases}$$

$$pf = \int_{\Omega} f dp = \int_{\Omega} (\lambda_1 I_{\omega_1} + \lambda_2 I_{\omega_2} + \cdots + \lambda_m I_{\omega_m}) dp = \sum_{i=1}^m \lambda_i \times p(\omega_i).$$

Analogously,

$$p_n f = \sum_{i=1}^m \lambda_i \times p_{n_i}(\omega_i).$$

By virtue of the conditions of the theorem,

$$\lim_{k \rightarrow \infty} p_{n_k}(\omega_i) = p(\omega_i) \quad \forall \omega_i \in \Omega,$$

and hence  $p_{n_k} f \rightarrow pf$  as  $k \rightarrow \infty$ , i.e.,  $p$  is the limit point of the subsequence  $\{p_{n_k}\}$ ; hence  $p \in P(\theta)$ . The theorem is proved.

Using this theorem, one can construct the statistical regularity of a sequence when the quantity of elements in the set  $\Theta$  is “not too big.” In this case one can select from the sequence  $\{p_n\}$  all subsequences  $\lambda$  such that for any of them and for any subset  $A$  from the set  $\Omega$  the limit

$$\lim_{k \rightarrow \infty} p_{n_k}(A) \tag{4.5}$$

exists. It follows from Theorem 4.4 that  $P(\bar{\omega})$  coincides with the set of measures that are the limits of these subsequences.

*Example 4.1.* Consider the following sequence of elements of the set  $\Omega = \{0, 1\}$ :

010011000011110000000011111111 000000000000000111111111111111 0000....

The sequence is built according to the following rule: the sequence consists of series of zeros and ones, a series of ones following a series of zeros, and every series of ones has length equal to the length of the previous series of zeros. The number of zeros in this previous series is equal to the number of zeros and ones in the previous series. The first series consists of one zero.

It is easy to see that the frequencies of zeros in the sequence  $\{p_n(\{0\})\}$  as  $n \rightarrow \infty$  are situated in the following limits:

$$\frac{1}{2} \leq p_n(\{0\}) \leq \frac{2}{3},$$

while for the frequencies of ones, we have the relation

$$p_n(\{1\}) = 1 - p_n(\{0\}).$$

The sequence of frequencies of zeros  $\{p_n(\{0\})\}$  has the limit points

$$p_{m,k}^1(\{0\}) = \frac{2^m + k}{2 \times 2^m + k}, \quad p_{m,k}^2(\{0\}) = \frac{2^{m+1}}{3 \times 2^m + k},$$

$$m = 0, 1, 2, 2, \dots, \quad k \in [0, 2^m] \cup Z,$$

where  $Z$  is the set of integers; this means that any point from the interval  $[1/2, 2/3]$  is a limit point for the sequence of frequencies of zeros. The sequence of frequencies of ones  $\{p_n(\{1\})\}$  has the limit points

$$p_{m,k}^1(\{1\}) = 1 - p_{m,k}^1(\{0\}), \quad p_{m,k}^2(\{1\}) = 1 - p_{m,k}^2(\{0\}).$$

Taking into account that the set of limit points of zeros is the interval  $[\frac{1}{2}, \frac{2}{3}]$ , one can conclude that the statistical regularity of the sequence that we are considering in this example has the form

$$P(\bar{\theta}) = \left\{ P(\cdot)/P(0) \in \left[ \frac{1}{2}, \frac{2}{3} \right], \quad p(1) = 1 - P(0) \right\}.$$

#### 4.6 Statistical Regularity of $\Gamma_I$ -Independence

The definition of  $\Gamma_I$ -independence of two sequences was given in Chapter 3. Now using the notion of statistical regularity, one can formulate necessary and sufficient conditions of  $\Gamma_I$ -independence [90].

**Theorem 4.5.** *The sequences  $\bar{\theta} \in (\mathbb{N} \rightarrow X)$  and  $\bar{u} \in (\mathbb{N} \rightarrow Y)$  are  $\Gamma_I$ -independent if and only if their joint statistical regularity  $P(\bar{\theta}, \bar{u})$  possesses the property that for any  $\varepsilon > 0$  one can find finite subdivisions*

$$X = \sum_{i=I} X_i, \quad Y = \sum_{j=J} Y_j, \quad I \times J = E_0 + E_1$$

of sets  $X, Y$ , and  $I \times J$  into nonoverlapping subsets satisfying the following conditions:

##### Condition 1

$$\sup_{\substack{x_1, x_2 \in X_i \\ y_1, y_2 \in Y_j}} |l(x_1, y_1) - l(x_2, y_2)| < \varepsilon \quad \text{for all } (i, j) \in E_1;$$

**Condition 2** For any  $p_0 \in P(\bar{\theta}, \bar{u})$  and any  $(i, j) \in (I \times Y)$ ,

$$p_0(X_i \times Y_i) - p_0(X_i \times Y) \times p_0(X \times Y_i) = 0;$$

**Condition 3** For any  $p_0 \in P(\bar{\theta}, \bar{u})$ ,

$$\sum_{(i,j) \in E_0} p_0(X_i \times Y_j) < \varepsilon.$$

*Proof.* Necessity: Let the sequences  $\bar{\theta}$  and  $\bar{u}$  be  $\Gamma_l$ -independent. Show that in this case the conditions of Theorem 4.5 are satisfied. Condition 1 coincides with Condition 1 of the definition of  $\Gamma_l$ -independence. Consider Conditions 2 and 3 of Theorem 4.5. Fix arbitrary  $X_i$  and  $Y_j$  from the subdivisions of sets  $X$  and  $Y$  corresponding to some positive  $\varepsilon$ . Prove first Condition 2. It follows from Condition 2 of  $\Gamma_l$ -independence that

$$\lim \{p_n(X_i \times Y_j) - p_n(X_i \times Y) \times p_n(X \times Y_j)\} = 0, \quad (4.6)$$

where

$$p_n(X_i \times Y_j) = \frac{1}{n} \sum_{k=1}^n I_{X_i}(\theta_k) I_{Y_j}(u_k).$$

Consider now some  $p_0$  from  $P(\bar{\theta}, \bar{u})$ . Since  $p_0$  is the limit point of the sequence  $\{p_n\}$  in the topological space  $(PF(X \times Y), \tau(X \times Y))$ , it follows that for any neighborhood  $V(p_0)$  of the point  $p_0$  of the topological space  $(PF(X \times Y), \tau(X \times Y))$  and for any  $m_0 \in \mathbb{N}$ , there is  $n_0 \in \mathbb{N}$  such that  $p_{n_0} \in V(p_0)$  for  $n_0 > m_0$ .

Consider the sequence  $\left\{ U_{I_{X_i \times Y_j}, I_{X_i \times Y}, I_{X \times Y_j}}^r(p_0) \right\}_{r=1}^{\infty}$  of neighborhoods of the form

$$U_{I_{X_i \times Y_j}, I_{X_i \times Y}, I_{X \times Y_j}}^r(p_0) = \left\{ p' \in PF(X \times Y) : |pf - p'f| < \frac{1}{r}, \quad r \in \mathbb{N}, \quad f \in \{I_{X_i \times Y_j}, I_{X_i \times Y}, I_{X \times Y_j}\} \right\}.$$

Further, for convenience denote the neighborhood

$$U_{I_{X_i \times Y_j}, I_{X_i \times Y}, I_{X \times Y_j}}^r(p_0)$$

by  $U_{i,j}^r(p_0)$ .

Consider now a subsequence of natural numbers  $\{n_k\}$  such that

$$\begin{aligned} n_1 &\geq 1, & p_{n_1} &\in U_{i,j}^1(p_0), \\ n_2 &\geq n_1, & p_{n_2} &\in U_{i,j}^2(p_0), \\ &\dots & & \\ n_k &\geq n_{k-1}, & p_{n_k} &\in U_{i,j}^k(p_0), \\ &\dots & & \end{aligned}$$

By virtue of what was said above, such a sequence exists. Hence  $\forall k \in \mathbb{N}$  we have the inequalities

$$\begin{aligned} |p_{n_k}(X_i \times Y_j) - p_0(X_i \times Y_j)| &< \frac{1}{k}, \\ |p_{n_k}(X_i \times Y) - p_0(X_i \times Y)| &< \frac{1}{k}, \\ |p_{n_k}(X \times Y_j) - p_0(X \times Y_j)| &< \frac{1}{k}, \end{aligned}$$

and for the real numbers  $\{p_{n_k}(X_i \times Y_j)\}, \{p_{n_k}(X_i \times Y)\}, \{p_{n_k}(X \times Y_j)\}$  we have the equalities

$$\begin{aligned} \lim_{k \rightarrow \infty} p_{n_k}(X_i \times Y_j) &= p_0(X_i \times Y_j), \\ \lim_{k \rightarrow \infty} p_{n_k}(X_i \times Y) &= p_0(X_i \times Y), \\ \lim_{k \rightarrow \infty} p_{n_k}(X \times Y_j) &= p_0(X \times Y_j). \end{aligned}$$

Consequently,

$$\begin{aligned} \lim_{k \rightarrow \infty} \{ (X_i \times Y_j) - p_{n_k}(X_i \times Y) \times p_{n_k}(X \times Y_j) \} \\ = p_0(X_i \times Y_j) - p_0(X_i \times Y) \times p_0(X \times Y_j). \end{aligned}$$

By virtue of Condition 1,

$$\lim_{k \rightarrow \infty} \{ p_{n_k}(X_i \times Y_j) - p_{n_k}(X_i \times Y) \times p_{n_k}(X \times Y_j) \} = 0,$$

and hence

$$p_0(X_i \times Y_j) - p_0(X_i \times Y) \times p_0(X \times Y_j) = 0.$$

Condition 2 is proved.

Now consider Condition 3. Take again some  $p_0 \in P(\bar{\theta}, \bar{u})$  and the neighborhood of  $p_0$

$$\begin{aligned} W_{E_0}(p_0) \\ = \left\{ p' \in PF(X \times Y) : |pf - p'f| < \frac{1}{r}, r \in \mathbb{N}, f \in \{I_{X_i \times Y_j} : (i, j) \in E_0\} \right\}. \end{aligned}$$

Consider a sequence of natural numbers  $\{n_k\}$  such that  $p_{n_k} \in W_{E_0}^k(p_0)$ , which means that

$$|p_{n_k}(X_i \times Y_j) - p_0(X_i \times Y_j)| < \frac{1}{k} \quad \forall (i, j) \in E_0.$$

Hence

$$\lim_{k \rightarrow \infty} \sum_{(i,j) \in E_0} p_{n_k}(X_i \times Y_j) = \sum_{(i,j) \in E_0} p_0(X_i \times Y_j).$$

Therefore we have

$$\lim_{k \rightarrow \infty} \sum_{(i,j) \in E_0} p_{n_k}(X_i \times Y_j) = \sum_{(i,j) \in E_0} p_0(X_i \times Y_j).$$

By virtue of Condition 3 from Definition 3.1 of  $\Gamma$ -independence, we have

$$\lim_{k \rightarrow \infty} \sum_{(i,j) \in E_0} p_{n_k}(X_i \times Y_j) < \varepsilon,$$

i.e.,

$$\sum_{(i,j) \in E_0} p_0(X_i \times Y_j) < \varepsilon.$$

Condition 3 is proved.

**Sufficiency:** Suppose the conditions of the theorem are satisfied. Choose some  $\varepsilon > 0$ . Let  $\{X_i\}$  and  $\{Y_j\}$  be the corresponding subdivisions of  $X$  and  $Y$ . Condition 1 from the definition of  $\Gamma$ -independence is clearly satisfied.

We prove now that Conditions 2 and 3 of the definition of  $\Gamma$ -independence are satisfied. Fix some  $X_i$  and  $Y_j$ . Suppose that Condition 2 is not satisfied for  $X_i$  and  $Y_j$ . Then there is a sequence of natural numbers  $\{n_k\}$  such that

$$\lim_{k \rightarrow \infty} \left[ \frac{1}{n_k} \sum_{k=1}^{n_k} I_{X_i}(\theta_k) I_{Y_j}(u_k) - \frac{1}{n_k^2} \sum_{k=1}^{n_k} I_{X_i}(\theta_{k_1}) \times \sum_{k=1}^{n_k} I_{Y_j}(u_{k_2}) \right] = \alpha, \quad \alpha \neq 0. \quad (4.7)$$

Consider the sequence  $\{p_{n_k}\}$  corresponding to  $\{n_k\}$ . By virtue of the compactness of the topological space  $(PF(X \times Y), \tau(X \times Y))$ , the subsequence  $\{p_{n_k}\}$  has a nonempty set of limit points. Choosing any of them as  $p_0$ , we can build, as we did in the case of necessity, a subsequence  $\{P_{n_k}\}_{m=1}^{\infty}$  such that

$$\lim_{m \rightarrow \infty} p_{n_k}(X_i \times Y_j) = p_0(X_i \times Y_j),$$

and by virtue of (4.7),

$$p_0(X_i \times Y_j) - (p_0(X_i \times Y_j) \times p_0(X_i \times Y_j)) = \alpha,$$

but  $p_0 \in P(\bar{\theta}, \bar{u})$  and consequently

$$P_0(X_i \times Y_j) - p_0(X_i \times Y) \times p_0(X \times Y_j) = 0.$$

This contradiction proves that Condition 2 of the definition of  $\Gamma$ -independence is satisfied.

Condition 3 is proved analogously. Suppose that condition 3 is not satisfied, i.e.,

$$\limsup_{k \rightarrow \infty} \sum_{(i,j) \in E_0} \frac{1}{n} \sum_{m=1}^{n_k} I_{X_i}(\theta_m) I_{Y_j}(u_m) \geq \varepsilon.$$

Then one can select a subsequence  $\{n_k\}$  such that

$$\lim_{k \rightarrow \infty} \sum_{(i,j) \in E_0} \frac{1}{n} \sum_{m=1}^{n_k} I_{X_i}(\theta_m) I_{Y_j}(u_m) \geq \varepsilon,$$

i.e.,

$$\lim_{k \rightarrow \infty} \sum_{(i,j) \in E_0} p_{n_k}(X_i \times Y_j) \geq \varepsilon.$$

It follows that  $\exists p_0 \in P(\bar{\theta}, \bar{u})$  such that

$$\sum_{(i,j) \in E_0} p(X_i \times Y_j) \geq \varepsilon.$$

But this contradicts the condition of the theorem. Therefore our assumption is wrong, and Condition 3 is satisfied. The theorem is proved.

## 4.7 Sampling Directedness as a Realization of a Random-in-a-Broad-Sense Phenomenon

In this section the reader will find some alternative, probably more convenient, formulations and definitions of the concepts and notions proposed in the previous sections of this chapter [43].

Let us define what we understand as nonstochastic randomness. For this purpose let us, at first intuitively, agree what phenomena it is reasonable to consider as random. Their foremost difference from nonrandom phenomena consists in the fact that when we call a phenomenon random it always means that we do not know the regularities (call them local) that would allow us to predict the behavior of the phenomenon. The study of a random phenomenon can be twofold: one can reduce it to a nonrandom phenomenon, trying to discover its local regularities, and one can continue to consider it as random and try to understand only the global, statistical regularities, i.e., regularities of asymptotic behavior of the variables characterizing this phenomenon, such as frequencies of landing in certain subsets and arithmetic averages of certain functionals.

If with the increase of the number of trials all these averages tend to certain limits (and some other conditions are satisfied; see [47]), then such a phe-

nomenon is generally called stochastic. And probability theory, as is known, considers only such phenomena. In contrast, it is natural to consider as random in a broad sense those mass phenomena that are studied only to within their statistical regularities.

Recall that the simplest mathematical model of a mass phenomenon is an ordinary sequence. In order to obtain a model of a random phenomenon on the basis of such a sequence, one needs to identify the sequences that have equivalent statistical properties.

**Definition 4.4.** Let  $X$  be an arbitrary set. We shall say that two sequences  $\bar{x}^{(1)}$  and  $\bar{x}^{(2)}$  of elements of the set  $X$  are *statistically equivalent* ( $S$ -equivalent) if for any natural number  $m$  and any bounded mapping  $\gamma \in (X \rightarrow \mathbb{R}^m)$ , the set of limit points of the sequence

$$\left\{ \bar{y}_n^{(k)}; n \in \mathbb{N} \right\}, \quad \bar{y}_n^{(k)} = \frac{1}{n} \sum_{i=1}^n \gamma(\bar{x}_i^{(k)}),$$

does not depend on  $k \in \{1, 2\}$ .

The class of  $S$ -equivalence of the sequence  $\bar{x} \in X^{\mathbb{N}}$  will be denoted by  $S(\bar{x})$ .

Our first goal is to find the invariant of the relation of  $S$ -equivalence. To this end, we introduce several notions.

Let  $M$  be a Banach space of bounded real functions defined on the set  $X$ , let  $M^*$  be the dual space of the space  $M$ , and  $\tau$  a weak- $*$  topology in  $M^*$ . Further, let  $PF(X)$  be the subspace of the topological space  $(M^*, \tau)$  defined by the formula

$$PF(X) = \{p \in M^* : p(\mathbf{1}_A) = 1, p(f) \geq 0 \text{ for } f \geq 0\},$$

where  $\mathbf{1}_A(\cdot)$  is the characteristic function of the set  $A$ .

In what follows we shall often write  $p(A)$  instead of  $p(\mathbf{1}_A)$ , identifying the elements of the set  $PF(X)$  with the measures (finitely additive and normalized) on  $2^X$ .

Then obviously,  $p(f)$  will be simply the integral  $p(f) = \int f(x)p(dx)$ , defined naturally due to the boundedness of the function  $f$ .

Now to an arbitrary sequence  $\bar{x} = \{\bar{x}_n; n \in \mathbb{N}\} \in X^{\mathbb{N}}$  we associate the sequence of measures from  $PF(X)$  defined as

$$\left\{ \bar{p}_{\bar{x}}^{(n)}(\cdot); n \in \mathbb{N} \right\}, \quad \bar{p}_{\bar{x}}^{(n)}(A) = \frac{1}{n} \sum_{i=1}^n \mathbf{1}_A(\bar{x}_i), \quad \forall A \subseteq X.$$

Due to the compactness of the set  $PF(X)$  (as a bounded closed set in  $(M^*, \tau)$ ), the sequence  $\left\{ \bar{p}_{\bar{x}}^{(n)}(\cdot); n \in \mathbb{N} \right\}$  will have a nonempty closed set of limit points, which we shall denote by  $P_{\bar{x}}$  and call the *regularity* of this sequence. We therefore introduce the following definition.

**Definition 4.5.** We shall call any nonempty closed set of the space  $PF(X)$  a *regularity* on  $X$ . We shall denote by  $\mathbb{P}(X)$  the set of all regularities on  $X$  and to any sequence  $\bar{x} \in X^{\mathbb{N}}$  we shall associate its regularity. Finally, for  $m \in \mathbb{N}$ ,  $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_m) \in (X \rightarrow \mathbb{R}^m)$ , and  $P \in \mathbb{P}(X)$ , we shall denote by  $P(\gamma)$  the set

$$\{(r_1, r_2, \dots, r_m) \in \mathbb{R}^m : \exists p \in P \forall i \in \overline{1, m}, r_i = p(\gamma_i)\},$$

and in particular,  $p(\gamma) = (p(\gamma_1), p(\gamma_2), \dots, p(\gamma_m))$  for  $p \in PF(X)$ .

We have the following theorem.

**Theorem 4.6.** *The mapping  $\bar{x} \mapsto P_{\bar{x}}$  is the invariant of the relation of  $S$ -equivalence on  $X^{\mathbb{N}}$ .*

This statement will be proved below in a more general form. So far, however, let us agree to call the classes of  $S$ -equivalence of sequences the *simplest random phenomena*, and we shall call their regularities *statistical regularities* of the corresponding phenomena. Any sequence  $\bar{x} \in X^{\mathbb{N}}$  will be considered as a realization of a simplest random phenomenon  $S(\bar{x})$ .

The connection between the notions introduced above and probabilistic notions follows directly from the enforced law of large numbers.

**Theorem 4.7.** *Let  $X$  be a finite set,  $\mu$  a probability distribution on  $X$ , and  $\bar{\xi} = \{\bar{\xi}_n; n \in \mathbb{N}\}$  a sequence of independent (in the usual sense) random elements taking values in  $X$  with distribution  $\mu$ . Then with probability 1 the sequence  $\bar{x}$  of the values of the sequence  $\bar{\xi}$  will be a realization of the simplest random phenomenon with statistical regularity  $P_{\bar{x}} = \{\mu\}$ , that is, consisting of the single distribution  $\mu$ .*

When the set  $X$  is infinite, everything becomes much more complicated. In this case, the capabilities of sequences, generally speaking, are insufficient to guarantee that the frequencies of hitting all measurable sets will tend to their limits simultaneously. Moreover, it is easy to see that the regularities of sequences comprise only a small part of the set of all regularities on  $X$ , since they are concentrated only on a countable subset of the set  $X$ . This seems to reflect the fact that sequences comprise only a small part of all mass phenomena. It turns out that a more general notion of *sampling directedness* is, as we shall see later, already sufficient for our goals.

**Definition 4.6.** We shall define a *sampling directedness* (s.d.) in  $X$  to be any directedness  $\varphi = \{\varphi_\lambda, \lambda \in \Lambda, \geq\}$  taking values in the sampling space

$$X^\infty = \bigcup_{n=1}^{\infty} X^n.$$

If  $\lambda \in \Lambda$ ,  $\varphi_\lambda \in X^n$  then we set  $n = n_\lambda$ ,  $\varphi_\lambda = (\varphi_{\lambda 1}, \varphi_{\lambda 2}, \dots, \varphi_{\lambda n_\lambda})$  and associate to this  $\lambda$  the measure  $p_\varphi^{(\lambda)} \in PF(X)$  defined as

$$p_\varphi^{(\lambda)}(A) = \frac{1}{n_\lambda} \sum_{i=1}^{n_\lambda} \mathbf{1}_A(\varphi_{\lambda i}), \quad A \subseteq X.$$

We shall call the set  $P_\varphi$  of limit points of the directedness  $p_\varphi = \{p_\varphi^\lambda, \lambda \in \Lambda, \geq\}$  the *regularity of the sampling directedness*  $\varphi$ . We shall denote the class of all s.d. in  $X$  by  $\Phi(X)$ .

Now we can extend the relation of  $S$ -equivalence to the whole of  $\Phi(X)$ .

**Definition 4.7.** We shall consider sampling directednesses  $\varphi^{(k)} \in \Phi(X)$ ,  $k = 1, 2$ , as *S-equivalent* if for any natural number  $m$  and any bounded mapping  $\gamma \in (X \rightarrow \mathbb{R}^m)$  the set of limit points of the directedness of the averages

$$\left\{ y_\lambda^{(k)}, \lambda \in \Lambda, \geq \right\}, \quad y_\lambda^{(k)} = \frac{1}{n_\lambda} \sum_{i=1}^{n_\lambda} \gamma(\varphi_{\lambda i}^{(k)}), \quad (4.8)$$

does not depend on  $k \in \{1, 2\}$ .

We can now reformulate Theorem 4.2 in the following way.

**Theorem 4.8.** (i) *For any s.d.  $\varphi \in \Phi(X)$ , any natural number  $m$ , and any bounded mapping  $\gamma \in (X \rightarrow \mathbb{R}^m)$ , the set of limit points of the directedness (4.8) can be written as  $P_{\varphi^{(k)}}(\gamma)$ .*

(ii) *The mapping  $\varphi \mapsto P_\varphi$ , defined on  $\Phi(X)$ , is an invariant of the relation of  $S$ -equivalence.*

(iii) *This mapping is a mapping on the whole set  $\mathbb{P}(\mathbb{X})$ , i.e., the set  $\Phi(X)/S$  of classes of  $S$ -equivalence and the set  $\mathbb{P}(X)$  of regularities can be put into one-to-one correspondence.*

The proof may be found in Section 4.3 of this chapter. However, we reproduce it below in a somewhat more convenient form.

*Proof.* Denote the set of limit points of an arbitrary directedness

$$g = \{g_\alpha, \alpha \in A, \geq\}$$

by  $\text{LIM}(g)$  or  $\text{LIM}\{g_\alpha, \alpha \in A, \geq\}$ . Denote the set of bounded mappings from  $X$  into  $\mathbb{R}^m$  by  $M^m$ . We need to establish the following three facts:

- (1) The relation  $\text{LIM}\{y_\lambda, \lambda \in \Lambda, \geq\} = P_\varphi(\gamma)$  is true for all  $m \in \mathbb{N}$ ,  $\gamma \in M^m$ ,  $\varphi \in \Phi(X)$ .
- (2) If  $P_1, P_2 \in \mathbb{P}(X)$ ,  $P_1 \neq P_2$ , then there exist  $m \in \mathbb{N}$  and  $\gamma \in M^m$  such that  $P_1(\gamma) \neq P_2(\gamma)$ .

(3) For any regularity  $P \in \mathbb{P}(X)$  there exist s.d.  $\varphi \in \Phi(X)$  such that  $P = P_\varphi$ .

We begin with a proof of statement (1). Let  $r \in \text{LIM}(y)$ ,  $y = \{y_\lambda, \lambda \in \Lambda, \geq\}$ . Then there exist a subdirectedness of the directedness  $y$  converging to  $t$ , i.e., there exist (see [44]) a directed set  $(A, \geq)$  and a function  $f : A \rightarrow \Lambda$  such that the directedness  $\bar{y} = y \circ f$  converges to  $r$ , and in addition, for any  $\lambda \in \Lambda$  there exists  $\alpha_1 \in A$  such that  $f(\alpha) \geq \lambda$  for all  $\alpha \geq \alpha_1$ .

Consider now the directedness of measures  $\bar{p}_\varphi = p_\varphi \circ f$ , where

$$p_\varphi = \left\{ p_\varphi^{(\lambda)}, \lambda \in \Lambda, \geq \right\}.$$

By virtue of the compactness of the space  $(PF(X), \tau)$  it has at least one limit point. Denote it by  $p_0$  and consider a subdirectedness  $\bar{\bar{p}}_\varphi$  of the directedness  $\bar{p}_\varphi$  converging to  $p_0$ . Let it be  $\bar{\bar{p}}_\varphi = \bar{p}_\varphi \circ g = p_\varphi \circ f \circ g$ ,  $g : B \rightarrow A$ . Then we have that the directedness  $\bar{\bar{y}} = y \circ f \circ g$ , on the one hand, converges to  $r$ , and on the other hand,  $\bar{\bar{y}}_\beta = \bar{\bar{p}}_\varphi^{(\beta)}(\gamma)$ ,  $\beta \in B$ , so that

$$r = \lim_{\beta} \bar{\bar{p}}_\varphi^{(\beta)}(\gamma) = p_0(\gamma) \in P_\varphi(\gamma).$$

In the same manner it is proved that  $\text{LIM}(y) \subseteq P_\varphi(\gamma)$ .

Conversely, if  $p_0 \in P_\varphi$ ,  $r = p_0(\gamma)$ , then there exists a subdirectedness  $\tilde{p}_\varphi = \{\tilde{p}_\varphi^\alpha, \alpha \in A, \geq\}$  of the directedness  $p_\varphi$  converging to  $p_0$ . But in this case,

$$\lim_{\alpha} \tilde{p}_\varphi^{(\alpha)}(\gamma_i) = p_0(\gamma_i)$$

for all  $i \in \overline{1, m}$ , which means that  $\lim_{\alpha} \tilde{p}_\varphi^{(\alpha)}(\gamma) = p_0(\gamma)$ . And since  $\tilde{p}_\varphi^{(\alpha)}(\gamma) = y_\lambda$  for  $\lambda = f(\alpha)$ , this proves statement (1).

In order to prove (2), assume that there exists  $p_1 \in P_1 \setminus P_2$ . Since the set  $P_2$  is closed, there exists a neighborhood of the point  $p_1$  that does not intersect  $P_2$ , which means that there exist  $\varepsilon > 0$ ,  $\gamma_1, \gamma_2, \dots, \gamma_m \in M$  such that

$$\forall p_2 \in P_2, \quad \exists i \in \overline{1, m}, \quad |p_1(\gamma_i) - p_2(\gamma_i)| > \varepsilon,$$

so that if  $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_m)$ , then  $p_1(\gamma) \notin P_2(\gamma)$ .

The complete proof of (3) may be found in Section 4.3. Here we shall outline the main ideas of the proof. Let  $Q$  be the set of all measures  $q \in PF(X)$  such that each of them is concentrated on a finite set  $X_q \subseteq X$ , and in addition, all numbers  $q(x)$ ,  $x \in X_q$  are rational. One can show that the set  $Q$  is everywhere dense in  $(PF(X), \tau)$ .

Now to an arbitrary regularity  $P \in \mathbb{P}(X)$  we associate the directed set  $(\Lambda, \geq)$  such that

$$\Lambda = \mathbb{R}^+ \times M^\infty \times P, \quad \mathbb{R}^+ = ]0, \infty[, \quad M^\infty = \bigcup_{m=1}^{\infty} M^m,$$

and the relation ( $\geq$ ) is given by the formula

$$\begin{aligned} (\varepsilon_1, \gamma_{11}, \gamma_{12}, \dots, \gamma_{1n_1}, p_1) &\geq (\varepsilon_2, \gamma_{21}, \gamma_{22}, \dots, \gamma_{2n_2}, p_2) \\ &\Leftrightarrow (\varepsilon_1 \leq \varepsilon_2, \{\gamma_{11}, \gamma_{12}, \dots, \gamma_{1n_1}\} \supseteq \{\gamma_{21}, \gamma_{22}, \dots, \gamma_{2n_2}\}). \end{aligned}$$

Finally, to any  $\lambda = (\varepsilon, \gamma_1, \gamma_2, \dots, \gamma_m, p) \in \Lambda$  we associate some

$$q_\lambda \in \mathcal{Q} \cap \{p' \in PF(X) : \forall i \in \overline{1, m}, |p(\gamma_i) - p'(\gamma_i)| < \varepsilon\}.$$

It is proved further that to any  $\lambda \in \Lambda$  one can associate simultaneously a sequence of points  $x_1^{(\lambda)}, x_2^{(\lambda)}, \dots, x_{n_\lambda}^{(\lambda)} \in X_q$  satisfying the condition

$$q_\lambda(A) = \frac{1}{n_\lambda} \sum_{i=1}^{n_\lambda} \mathbf{1}_A(x_i^{(\lambda)}), \quad \forall A \subseteq X.$$

It remains to choose  $\varphi_\lambda = (x_1^{(\lambda)}, x_2^{(\lambda)}, \dots, x_{n_\lambda}^{(\lambda)})$ , and we obtain an s.d.  $\varphi : \lambda \mapsto \varphi_\lambda$  that has the regularity  $P_\varphi = P$ .

Therefore we can justly give the following definition.

**Definition 4.8.** Any class of  $S$ -equivalence of sampling directednesses in  $X$  is called a *random-in-a-broad-sense phenomenon in  $X$* . The regularity  $P_\varphi$  is called a *statistical regularity of the random phenomenon  $S(\varphi)$* . Any s.d.  $\varphi' \in S(\varphi)$  is called a *realization of the random phenomenon  $S(\varphi)$* . A random phenomenon having statistical regularity  $P$  is called  *$\mu$ -stochastic* if there exists a nontrivial  $\sigma$ -algebra  $\mathcal{A} \subseteq 2^X$  on which  $\mu$  is a  $\sigma$ -additive probability and  $p(A) = \mu(A)$  for all  $p \in P, A \in \mathcal{A}$ .



# Chapter 5

## General Decision Problems

I warne thee (quoth he), Icarus, a middle race to keepe.  
For if thou hold too low a gate, the dankenese of the deepe  
Will overlade thy wings with wet. And if thou mount too hie,  
The Sunne will sindge them. Therfore see betweene them both thou flie.

---

Daedalus to Icarus, Ovid, *Metamorphoses*

### 5.1 Preliminaries

The regularities on  $\Theta$  provide a mathematical tool that can be used for the description of mass events depending on a parameter and estimated on average. It is interesting, however, that this conclusion can be obtained regardless of the results of Chapter 4.

Let  $\mathbb{Z}$  be the class of all ordered triples of the form

$$Z = (\Theta, U, L),$$

where  $\Theta$  and  $U$  are arbitrary nonempty sets, and

$$L : \Theta \times U \rightarrow \mathbb{R}$$

is a real bounded function. Recall that we call any such triple a *matrix scheme of a decision situation* or simply a *matrix decision scheme* and interpret it in the following way:  $\Theta$  is a set of possible values of the unknown parameter  $\theta$ ,  $U$  a set of possible (“our”) decisions, and  $L$  a loss function. We denote by  $\mathbb{Z}(\Theta)$  the subclass of all decision schemes of the form

$$Z = (\Theta, \cdot, \cdot),$$

where  $\Theta$  is a fixed set. Note that in this chapter we shall use only matrix schemes of decision situations.

In order to choose the optimal decision from the set  $U$  of all possible decisions it is not enough to know only the decision scheme. One needs to know how for a given decision scheme one can construct some preference relation on  $U$ . There are convincing reasons (see Section 5.2) to require that this preference relation be given by “its” own loss function

$$L_Z^* : U \rightarrow \mathbb{R},$$

which we call a “criterion.” For example, if  $\theta$  is random with a given distribution, one usually prefers the decision that leads to smaller average losses (Bayesian criterion). If nothing is known about  $\theta$ , then decisions are, as a rule, compared with respect to the maximal losses (minimax criterion), and so on. However, what is it precisely that dictates such a choice of criterion? Is it possible to state that as soon as the decision scheme and the information about the behavior of  $\theta$  are fixed, the criterion choice is thereby already predetermined?

As is known (see, for instance, [58]), in the general case that is not so. Therefore, some other considerations, the nature of which is not quite clear yet, participate in the criterion choice. But the necessity for clarity in this question is obvious, for arbitrariness in the criterion choice is, in essence, the arbitrariness in choice of a decision, which, as a result, turns out to be weakly connected with the accepted mathematical model.

Our aim in this chapter is to isolate the class of criterion choice rules such that inside this class a criterion is already a univalent function of the decision scheme and the “information about  $\theta$ ” [37, 38].

**Definition 5.1.** We define a *criterion choice rule* to be any mapping  $\pi$ , defined on  $\mathbb{Z}(\Theta)$  and associating to every scheme  $Z = (\Theta, U, L)$  some real function  $L_Z^*(\cdot)$ , a criterion, determined on  $U$ . We denote the class of all criterion choice rules by  $\Pi(\Theta)$  and include in the subclass  $\Pi_1(\Theta) \subset \Pi(\Theta)$  all criterion choice rules that satisfy the following three conditions:

**Condition 1** If  $Z_i = (\Theta, U_i, L_i) \in \mathbb{Z}(\Theta)$ ,  $i = 1, 2$ ,  $U_1 \subset U_2$ , and  $L_1(\theta, u) = L_2(\theta, u) \forall u \in U_1, \forall \theta \in \Theta$ , then  $L_{Z_1}^*(u) = L_{Z_2}^*(u)$  for all  $u \in U_1$ .

**Condition 2** If  $Z = (\Theta, U, L) \in \mathbb{Z}(\Theta)$ ,  $u_1, u_2 \in U$ , then if

$$L(\theta, u_1) \leq L(\theta, u_2)$$

is true for all  $\theta \in \Theta$ , then

$$L_Z^*(u_1) \leq L_Z^*(u_2),$$

and if  $a, b \in \mathbb{R}$ ,  $a \geq 0$ , then if

$$L(\theta, u_1) = aL(\theta, u_2) + b$$

is true for all  $\theta \in \Theta$ , then

$$L_Z^*(u_1) = aL_Z^*(u_2) + b.$$

**Condition 3** If  $Z = (\Theta, U, L) \in \mathbb{Z}(\Theta)$ ,  $u_1, u_2, u_3 \in U$ , and

$$L(\theta, u_1) + L(\theta, u_2) = 2L(\theta, u_3), \quad \forall \theta \in \Theta, \quad (5.1)$$

then we have the inequality

$$L_Z^*(u_1) + L_Z^*(u_2) \geq 2L_Z^*(u_3). \quad (5.2)$$

The first two conditions are quite natural: they are conditions imposed by the decision-maker on the criterion when there is no uncertainty (see Section 2.6). Therefore we concentrate only on the third condition, which characterizes the behavior of the decision-maker under uncertainty.

Let us compare two decision systems  $Z = (\Theta, U, L)$  and  $\tilde{Z} = (\tilde{\Theta}, \tilde{U}, \tilde{L})$ , where  $\tilde{\Theta} = \Theta \times \Theta$ ,  $\tilde{U} = U \times U$ ,  $\tilde{L}((\theta_1, \theta_2), (u_1, u_2)) = L(\theta_1, u_1) + L(\theta_2, u_2)$ . Interpreting  $\tilde{Z}$  as a two-stage decision problem under the same conditions (assigned by decision scheme  $Z$ ), it is natural to assume that

$$L_Z^*(u_1, u_2) = L_Z^*(u_1) + L_Z^*(u_2).$$

Thus inequality (5.2) means that according to the assumption (5.1), it is better to choose  $u_3$  twice than to choose  $u_1$  once and then  $u_2$ . What is the reason for this? We have from (5.1) that

$$\begin{aligned} & [L(\theta_1, u_3) + L(\theta_2, u_3)] - [L(\theta_1, u_1) + L(\theta_2, u_2)] \\ &= [L(\theta_2, u_1) + L(\theta_1, u_2)] - [L(\theta_2, u_3) + L(\theta_1, u_3)]. \end{aligned}$$

This may be interpreted as follows: for any pair of values  $\theta_1$  and  $\theta_2$ , the losses in system  $Z$  associated with the choice  $(u_3, u_3)$  do not depend on the order of appearance of  $\theta_1$  and  $\theta_2$ , but if we chose  $(u_1, u_2)$ , then the corresponding losses do depend on the order of appearance of  $\theta_1$  and  $\theta_2$ : if  $\theta_2$  follows  $\theta_1$ , the losses for  $(u_1, u_2)$  are smaller than for  $(u_3, u_3)$ , and in the other case, when  $\theta_1$  follows  $\theta_2$ , the losses for  $(u_1, u_2)$  are bigger than for  $(u_3, u_3)$ , but the difference is the same. Thus Condition 3 is just a specific form of the guaranteed result principle for mass events, which seems to be the most natural form of uncertainty aversion.<sup>1</sup>

<sup>1</sup> Perhaps numerous examples from the financial industry may serve as illustrations of this principle. Imagine that an investor chooses two stocks in which to place his wealth. He can invest all of it in one of the stocks or in both of them. In this case, the decision is the way the investor partitions his wealth,  $\{u\} = \{w_1, w_2\}$ ,  $w_1 + w_2 = 1$ . The parameter  $\theta$  is the market movement,  $\{\theta\} = \{r_1, r_2\}$ , where  $r_1$  and

Now introduce into consideration the mapping

$$\kappa : \mathbf{P}(\Theta) \rightarrow \Pi(\Theta)$$

defined in the following way. If  $\mathbf{P}(\Theta)$  is the family of all statistical regularities on  $\Theta$  (see Chapter 4), and if

$$P \in \mathbf{P}(\Theta), \quad \pi = \kappa(P), \quad Z = (\Theta, U, L) \in \mathbb{Z}(\Theta), \quad \pi(Z) = L_Z^*(\cdot),$$

then

$$L_Z^*(u) = \max_{p \in P} \int L(\theta, u) p(d\theta) \quad (5.3)$$

for all  $u \in U$  (the existence of a maximum follows from the set  $P(\Theta)$  being closed).

It turns out that for any decision scheme  $Z \in \mathbb{Z}(\Theta)$  and any statistical regularity  $P \in \mathbf{P}(\Theta)$  the criterion choice rule  $\pi = \kappa(P)$  belongs to the class  $\Pi_1(\mathbb{Z}(\Theta))$ . And conversely, for any criterion choice rule  $\pi$  from the class  $\Pi_1(\mathbb{Z}(\Theta))$  one can find a decision scheme  $Z \in \mathbb{Z}(\Theta)$  and a statistical regularity  $P \in \mathbf{P}(\Theta)$  such that  $\kappa(P) = \pi$ . In short, (Theorem 5.2),

$$\kappa_{\mathbb{Z}(\Theta)}(\mathbf{P}(\Theta)) = \Pi_1(\mathbb{Z}(\Theta)).$$

We have obtained thus that if we choose the guaranteed result principle (in the form of Condition 3) as the optimality principle, then for rather wide assumptions, in order to give an exact mathematical formulation of the decision problem once the decision scheme  $Z = (\Theta, U, L)$  is given, it is sufficient to fix only some statistical regularity  $P$  on  $\Theta$ .

In other words, the projector  $\pi$  belongs to the class  $\Pi_1$  if and only if it possesses the structure of mapping  $\kappa$ .

*Remark 5.1.* According to the terminology of Chapter 2, the pair  $S = (Z, P)$  is called a model of a decision situation, whereas the criterion choice rule, the projector,  $\pi$  is a model of a decision-maker. The pair  $(S, \pi)$  constitutes a model of a decision system. Now, according to Theorem 5.2, when Conditions 1–3 are

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$r_2$  are the stock returns. The loss function in this case is written as  $L(\theta, u) = w_1 \cdot r_1 + w_2 \cdot r_2$ , and is the “profit and loss” function. Let decision  $u_1$  consist in placing all the wealth in the first stock,  $u_1 = \{1, 0\}$ , while decision  $u_2$  is placing all the wealth in the second stock,  $u_2 = \{0, 1\}$ , and decision  $u_3$  consists in placing a half in the first and a half in the second,  $u_3 = \{\frac{1}{2}, \frac{1}{2}\}$ . It is clear that regardless of the market movement  $\theta = \{r_1, r_2\}$ , Condition 5.1 is satisfied in this case,  $r_1 + r_2 = 2(\frac{1}{2}r_1 + \frac{1}{2}r_2) = r_1 + r_2$  or  $L(\theta, u_1) + L(\theta, u_2) = 2L(\theta, u_3)$ . So if our investor belongs to the class  $\Pi_1$  of decision-makers, he would prefer to choose decision  $u_3$ . In other words, he would prefer to invest in the diversified, or market neutral, portfolio rather than in a directional one; he would prefer not to take directional risks. Moreover, as will become clear from Theorem 5.2, only for such investors does it make sense to use criterion (5.3), which becomes the usual criterion of expected losses, or expected utility, in the stochastic case.

satisfied, the decision system  $(S, \pi)$  induces the unique criterion  $L_Z^*$ . Therefore if we remain in the limits of the class  $\Pi_1$ , there is no difference in using  $(S, \pi)$ ,  $(S, L_Z^*)$ , or just  $S$ .

This allows us to introduce the following definition.

**Definition 5.2.** An ordered quadruple of the form  $S = (\Theta, U, L, P)$ , where  $Z = (\Theta, U, L) \in \mathbb{Z}$ , and  $P \in \mathbf{P}(\Theta)$ , is called a *general decision problem*. We denote the class of all decision problems of the form  $(\Theta, \cdot, \cdot, \cdot)$  by  $\mathbf{S}(\Theta)$ , and to any decision problem  $S = (\Theta, U, L, P) \in \mathbf{S}(\Theta)$  we associate the function (decision criterion)

$$L_Z^* : U \rightarrow \mathbb{R}, \quad L_Z^*(u) = \max_{p \in P} \int L(\theta, u) p(d\theta),$$

and the number

$$\rho(S) = \inf_{u \in U} L_Z^*(u),$$

called the *risk of the decision problem S*.

We call the decision problem  $S = (\Theta, U, L, P)$  a *Bayesian decision problem* if there is  $F \subseteq M(X)$  such that

- (1)  $L(\cdot, u) \in F$  for all  $u \in U$ ;
- (2) the regularity  $P$  in  $S$  is  $\mu$ - $F$ -stochastic (see Chapter 4).

We call a decision problem  $S = (\Theta, U, L, P)$  a *minimax decision problem* if  $P = PF(\Theta)$ .

It is easy to see that the criteria of these decision problems, in the sense of the above definition, correspond to Bayesian and minimax criteria in a generally accepted sense.

*Remark 5.2.* Note that by virtue of Theorem 4.8 (or Theorem 4.2) and Definition 4.8, to any regularity  $P$  one can associate some sampling directedness. But any sampling directedness is a realization of some random-in-a-broad-sense phenomenon. Therefore, criterion (5.3) makes sense only for multiple choices of decisions in the same situation, i.e., for a mass phenomenon.

## 5.2 On the Preference Relation in Mass Operations

In any decision system we deal with two sets of alternatives and preference relations on them: the set of consequences  $C$  or  $\Theta \times U$ , and the set of decisions  $U$ . We suppose that when the decision system contains uncertainty, the preference relation  $\beta_C$  on  $C$ , or on  $\Theta \times U$ , is given initially, but we ourselves have to construct the preference relation  $\beta_U$  on  $U$ .

In many practically significant cases the relation  $\beta_C$  is given by means of a loss function, i.e., it is supposed that

$$(\theta_1, u_1)\beta_C(\theta_2, u_2) \Leftrightarrow L(\theta_1, u_1) \leq L(\theta_2, u_2)$$

(here and below,  $\beta$  is interpreted as “not worse”). And generally speaking, it appears that the relation  $\beta_U$  has also to be defined by means of its own loss function  $L^*$ . However, there are in fact quite distinct intuitive prerequisites if we are to see in this assumption something more significant. It turns out that the condition that the preference relation is equivalent to its loss function is connected directly with the assumption of the possibility of comparing not only various alternatives but also arbitrary finite sets of all possible alternatives. Therefore, it is natural to conclude that if the loss function  $L$  exists, then the loss function  $L^*$  exists as well.

Note that in utility theory there are many results of this kind [20]. However, among them we could not find any to which we could simply refer, and therefore this result is presented here with a complete proof.

Let  $A$  be an arbitrary set. We shall use the following, basically standard, terminology and notation (see Appendix A.1). If  $\beta$  is a relation on  $A$ , i.e.,  $\beta \subseteq A \times A = A^2$ , then we consider the records  $(a, b) \in \beta$ ,  $a\beta b$ , and  $b\beta^{-1}a$  as equivalent, thus denoting by  $\beta^{-1}$  the inverse relation

$$\beta^{-1} = \{(a, b) \in A^2 : (b, a) \in \beta\}.$$

By definition,

$$\beta^0 = \beta \cap \beta^{-1}, \quad \beta' = \beta\beta^{-1} = \beta\beta^0, \quad A^{(\infty)} = \bigcup_{n=1}^{\infty} A^n,$$

and instead of  $(a\beta b) \wedge (b\beta c)$  we write simply  $a\beta b\beta c$ .

Next, if

$$\underline{a} = (a_1, a_2, \dots, a_k) \in A^{(\infty)}, \quad \underline{b} = (b_1, b_2, \dots, b_m) \in A^{(\infty)},$$

then define

$$\overline{a, \underline{b}} = (a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_m), \quad n(\underline{a}) = k, \quad n(\underline{b}) = m, \text{ etc.}$$

**Definition 5.3.** The relation  $\beta$  on  $A^{(\infty)}$  is called a *statistical preference* on  $A$  if it possesses the following properties:

**Property 1**  $\underline{a}\beta\underline{c}$  follows from  $\underline{a}\beta\underline{b}\beta\underline{c}$  (transitivity).

**Property 2**  $\beta \cup \beta^{-1} = A^{(\infty)}$ .

**Property 3** If  $\underline{a}, \underline{b} \in A^{(\infty)}$ ,  $n(\underline{a}) = n(\underline{b}) = n$ ,  $\underline{a} = (a_1, a_2, \dots, a_n)$ ,  $\underline{b} = (b_1, b_2, \dots, b_n)$ , and  $b_k = a_{ik}$  ( $1 \leq k \leq n$ ), where

$$\begin{pmatrix} 1 & 2 & \dots & n \\ i_1 & i_2 & \dots & i_n \end{pmatrix}$$

is some permutation, then  $\underline{a}\beta^0\underline{b}$ .

**Property 4** If  $\underline{a}, \underline{b}, \underline{c}, \underline{d} \in A^{(\infty)}$  and  $\underline{a}\beta^0\underline{b}$ , then  $\overline{\underline{c}\underline{a}\underline{b}\underline{d}} \Leftrightarrow \underline{c}\underline{\beta}\underline{d}$ .

**Property 5** If  $\underline{a}, \underline{b}, \underline{c}, \underline{d} \in A^{(\infty)}$ , and  $\underline{a}\beta'\underline{b}$ , then there exists a natural number  $n$  such that

$$\overbrace{\underline{a}, \underline{a}, \dots, \underline{a}}^n \underline{c} \beta' \overbrace{\underline{b}, \underline{b}, \dots, \underline{b}}^n \underline{d}.$$

**Theorem 5.1.** In order for  $\beta$  to be a statistical preference on  $A$  it is necessary and sufficient that there exist a function  $L : A \rightarrow R$  such that

$$\underline{a}\beta\underline{b} \Leftrightarrow \sum_{i=1}^{n(a)} L(a_i) \leq \sum_{j=1}^{n(b)} L(b_j)$$

for all  $\underline{a} = (a_1, a_2, \dots, a_{n(a)})$ ,  $\underline{b} = (b_1, b_2, \dots, b_{n(b)}) \in A^{(\infty)}$ .

This function is determined uniquely up to a scale factor; i.e., if  $L_1$  is a second function of this kind, then there exists  $r > 0$  such that

$$L_1(a) = rL(a) \quad \text{for any } a \in A.$$

*Proof.* (1) We determine first of all that the relation  $\beta^0$  is an equivalence relation on  $A^{(\infty)}$ , i.e., that it is symmetric, reflexive, and transitive.

- (i) Symmetry. If  $\underline{a}\beta^0\underline{b}$ , then  $\underline{a}\beta\underline{b}$  and  $\underline{b}\beta\underline{a}$ . But then also  $\underline{b}\beta^0\underline{a}$ .
- (ii) Reflexivity. Obviously,  $\underline{a}\beta\underline{a}$ , for otherwise  $(\underline{a}, \underline{a}) \notin \underline{\beta} \cup \underline{\beta}^{-1}$ , contrary to Property 2 from Definition 5.3. From here,  $\underline{a}\beta^{-1}\underline{a}$ , and thus,  $\underline{a}\beta^0\underline{a}$ .
- (iii) Transitivity. If  $\underline{a}\beta^0\underline{b}\beta^0\underline{c}$ , then  $\underline{a}\beta\underline{b}\beta\underline{c}$ , and by virtue of Property 1 from Definition 5.3,  $\underline{a}\beta\underline{c}$ . In precisely the same way we obtain that  $\underline{c}\beta\underline{a}$ , and thus  $\underline{a}\beta^0\underline{c}$ .

(2) We set

$$\beta^0(\underline{a}) = \{\underline{b} \in A^{(\infty)} : \underline{b}\beta^0\underline{a}\}, \quad X = \{x : \exists \underline{a} \in A^{(\infty)}, x = \beta^0(\underline{a})\}.$$

In other words,  $X$  is a factor set of the set  $A^{(\infty)}$  with respect to the equivalence relation  $\beta^0$ .

We set further

$$(\preceq) = \{(x, y) \in X^2 : \exists \underline{a} \in x, \underline{b} \in y (\underline{a}\beta\underline{b})\}$$

and prove that  $\preceq$  is a linear order on  $X$ , i.e.,

- (i)  $x \preceq z$  follows from  $x \preceq y \preceq z$ ;
- (ii)  $x = y$  follows from  $x \preceq y, y \preceq x$ ;
- (iii)  $(\preceq) \cup (\preceq)^{-1} = X^2$ .

For this purpose we verify first that

$$(\preceq) = \{(x, y) \in X^2; \forall \underline{a}' \in x, \underline{b}' \in y(\underline{a}'\beta\underline{b}')\}.$$

By definition we have  $\underline{a}\beta\underline{b}$ , and at the same time,  $\underline{a}\beta^0\underline{a}'$ , and so  $\underline{a}'\beta\underline{a}\beta\underline{b}\beta\underline{b}'$ , i.e.,  $\underline{a}'\beta\underline{b}'$ , as required.

Now we can prove the above statements:

- (i) Let  $a \in x, b \in y, c \in z$ . Then  $\underline{a}\beta\underline{b}\beta\underline{c}$ , and thus  $\underline{a}\beta\underline{c}$ , i.e.,  $x \preceq z$ .
- (ii) If  $x \preceq y, y \preceq x, \underline{a} \in x, \underline{b} \in y$ , then  $\underline{a}\beta\underline{b}\beta\underline{a}$ , i.e.,  $\underline{a}\beta^0\underline{b}$  and so  $\beta^0(a) = \beta^0(b) = x = y$ .
- (iii) This assertion follows immediately from Property 2.

(3) Let

$$\underline{a}, \underline{b} \in A^{(\infty)}, \quad x = \beta^0(a), \quad y = \beta^0(b), \quad z = \beta^0(\overline{\underline{a}, \underline{b}}).$$

We show that if  $\underline{a}' \in x, \underline{b}' \in y$ , then  $\underline{a}', \underline{b}' \in z$ , i.e., in essence,  $z$  is determined only by  $x$  and  $y$ , and the described construction defines a binary operation on  $X : x + y = z$ . To this end, note that  $\underline{a}'\beta^0\underline{a}, \underline{b}'\beta^0\underline{b}$ , and so by Property 4, taking into account Property 3, we have

$$\overline{\underline{a}'\beta^0\underline{a}, \underline{b}'\beta^0\underline{b}}, \quad \text{i.e., } \underline{a}', \underline{b}' \in z.$$

(4) We show that  $(X, +)$  is an abelian semigroup with a reduction law, i.e., that

- (i)  $x + y = y + x$  (commutativity);
- (ii)  $(x + y) + z = x + (y + z)$  (associativity);
- (iii)  $x + z = y + z \Rightarrow x = y$  (reduction rule);

To prove (i), note that if  $\underline{a} \in x, \underline{b} \in y$ , then  $(x + y) \ni \overline{\underline{a}, \underline{b}}\beta^0\overline{\underline{b}\underline{a}} \in (y + x)$ , and so  $x + y = y + x$ . Property (ii) follows from the fact that if  $\underline{a} \in x, \underline{b} \in y, \underline{c} \in z$ , then  $\overline{\underline{a}, \underline{b}, \underline{c}} = \overline{\underline{a}, \underline{b}, \underline{c}} = \overline{\underline{a}, \underline{b}, \underline{c}}$ . Finally, property (iii) follows from Property 4 in combination with Property 3: if  $\underline{a} \in x, \underline{b} \in y, \underline{c} \in z$ , then

$$\overline{\underline{a}, \underline{c}}\beta\overline{\underline{b}, \underline{c}} \Rightarrow \underline{a}\beta\underline{b} \Rightarrow x \preceq y$$

and

$$\overline{\underline{b}, \underline{c}}\beta\overline{\underline{a}, \underline{c}} \Rightarrow \underline{b}\beta\underline{a} \Rightarrow y \preceq x, \tag{5.4}$$

and since in our case  $\overline{\underline{a}\underline{c}}\beta^0\overline{\underline{b}\underline{c}}$ , we have  $\underline{a}\beta^0\underline{b}$ , i.e.,  $x = y$ .

(5) We show that the linear order  $\preceq$  on  $X$  is consistent with the semigroup operation  $(+)$  in the natural sense:

$$x \preceq y \Leftrightarrow x + z \preceq y + z \quad \forall x, y, z \in X.$$

Let

$$x \preceq y, \quad \underline{a} \in x, \quad \underline{b} \in y, \quad \underline{c} \in z.$$

Then by virtue of Property 4, we obtain

$$\underline{a}\underline{b}\underline{b} \Rightarrow \overline{\underline{a}, \underline{c}\underline{b}\underline{b}, \underline{c}} \Rightarrow x + z \preceq y + z.$$

We have already proved the reverse implication (5.4).

(6) We now make use of the theorem (see [51, Chapter II, 5, paragraph 3]) that any abelian semigroup with reduction law can be isomorphically embedded in an abelian group, and having performed the embedding  $(X, +)$  in the group  $(G, +)$ , we identify the elements  $x \in X$  with their images in this embedding.

As follows from the proof of this theorem, any element  $g \in G$  will then be representable in the form

$$g = x - y \quad (x, y \in X).$$

(7) We now extend the relation  $\preceq$  defined on  $X$  onto the whole set  $G$ , setting

$$\begin{aligned} (g_1 \preceq^* g_2) &\Leftrightarrow \exists x_1, y_1, x_2, y_2 \in X \\ &(g_1 = x_1 - y_1, g_2 = x_2 - y_2, x_1 + y_2 \preceq x_2 + y_1), \end{aligned}$$

and prove that  $\preceq^*$  is a linear order consistent with the group operation, i.e., that

- (i)  $\preceq^* \cup (\preceq^*)^{-1} = G$ ;
- (ii)  $\preceq^* \cap (\preceq^*)^{-1} = (=)$ ;
- (iii)  $g_1 \preceq^* g_2 \preceq g_3 \Rightarrow g_1 \preceq^* g_3$ ;
- (iv)  $g_1 \preceq^* g_2 \Leftrightarrow g_1 + g \preceq^* g_2 + g \quad \forall g_1, g_2, g \in G$ .

Relation (i) is obvious, since for any  $g_1, g_2 \in G$ , according to the proof of (6), there exist points  $x_1, x'_1, x_2, x'_2 \in X$  such that

$$x_1 - x'_1 = g_1, \quad x_2 - x'_2 = g_2,$$

and since

$$\preceq U \preceq^{-1} = X,$$

at least one of the relations

$$x_1 + x'_2 \preceq x_2 + x'_1 \quad \text{and} \quad x_2 + x'_1 \preceq x_1 + x'_2$$

must hold.

It is more convenient to prove (iv) first. To do this, having assumed that  $g_1 \preceq^* g_2$ , i.e., that there exist  $x_1, x'_1, x_2, x'_2 \in X$  such that

$$x_1 - x'_1 = g_1, \quad x_2 - x'_2 = g_2, \quad x_2 + x'_2 \preceq x_2 + x'_1, \quad (5.5)$$

we represent  $g$  in the form

$$g = \bar{x} - \bar{x}'$$

and set

$$\begin{aligned} x_1 + \bar{x} &= x_1^*, & x_2 + \bar{x} &= x_2^*, & x'_1 + \bar{x}' &= x'^*_1, & x'_2 + \bar{x}' &= x'^*_2, \\ g_1 + g &= g_1^*, & g_2 + g &= g_2^*. \end{aligned}$$

We obtain

$$\begin{aligned} x_1 - x'_1 + g &= x_1^* - x'^*_1 = g_1^*, \\ x_2 - x'_2 + g &= x_2^* - x'^*_2 = g_2^*, \\ x_1^* + x'^*_2 &= x_1 + \bar{x} + x'_2 + \bar{x}', \\ x_2^* + x'^*_1 &= x_2 + \bar{x} + x'_1 + \bar{x}', \end{aligned}$$

and by virtue of assumption (5.5), on the basis of the proof of item (5) above, we have

$$(x_1 + x'_2) + (\bar{x} + \bar{x}') \preceq (x_2 + x'_1) + (\bar{x} + \bar{x}'),$$

or on the other hand,

$$(x_1 + \bar{x}) + (x'_2 + \bar{x}') \preceq (x_2 + \bar{x})(x'_1 + \bar{x}'),$$

i.e.,

$$x_1^* + x'^*_2 \preceq x_2^* + x'^*_1,$$

and, by definition of the relation ( $\preceq^*$ ),  $g_1^* \preceq^* g_2^*$ . We obtain the inverse implication immediately by substituting  $g$  with  $(-g)$ . Relation (iv) is proved.

Incidentally we note that

$$g_1 \preceq^* g_2 \Leftrightarrow g_1 - g_2 \preceq^* 0 \Leftrightarrow 0 \preceq^* g_2 - g_1 \quad (5.6)$$

follows at once from the proved relation (iv).

Now we have to make sure that if  $g_1 \preceq^* g_2$ , then the relation

$$x_1 + x'_2 \preceq x_2 + x'_1$$

is valid for all  $x_1, x_2, x'_1, x'_2 \in X$  such that  $g_1 = x_1 - x'_1$ ,  $g_2 = x_2 - x'_2$ . Indeed, it follows from  $g_1 \preceq^* g_2$  that for some

$$\bar{x}_1, \bar{x}_2, \bar{x}'_1, \bar{x}'_2 \in X$$

the relations

$$g_1 = \bar{x}_1 - \bar{x}'_1, \quad g_2 = \bar{x}_2 - \bar{x}'_2, \quad \bar{x}_1 + \bar{x}'_1 \preceq \bar{x}'_2 + \bar{x}'_1 \quad (5.7)$$

are valid. From these relations we obtain

$$\bar{x}_1 - \bar{x}'_1 = x_1 - x'_1, \quad \bar{x}_2 - \bar{x}'_2 = x_2 - x'_2,$$

and so

$$x - \bar{x}_1 = x'_1 - \bar{x}'_1, \quad x'_2 - \bar{x}'_2 = x_2 - \bar{x}_2,$$

i.e.,

$$x_1 + x'_2 - \bar{x}_1 - \bar{x}'_2 = x'_1 + x_2 - \bar{x}'_1 - \bar{x}_2.$$

Now, denoting the common value of the right and left parts of the last relation by  $g$ , we note that

$$\begin{aligned} x_1 + x'_2 &= \bar{x}_1 + \bar{x}'_2 + x_1 + x'_2 - \bar{x}_1 - \bar{x}'_2 = \bar{x}_1 + \bar{x}'_2 + g, \\ x_2 + x'_1 &= \bar{x}_2 + \bar{x}'_1 + x_2 + x'_1 - \bar{x}_2 - \bar{x}'_1 = \bar{x}_2 + \bar{x}'_1 + g, \end{aligned}$$

and so, according to (5.6) and (5.7),

$$x_1 + x'_2 \preceq x_2 + x'_1,$$

which is what had to be proved.

We now prove (ii). If

$$g_1 \preceq^* g_2 \quad \text{and} \quad g_2 \preceq^* g_1$$

then, by virtue of what was proved, there exist  $x_1, x_2, x'_1, x'_2 \in X$  such that

$$x_1 + x'_2 \preceq x_2 + x'_1, \quad x_2 + x'_1 \preceq x_1 + x'_2,$$

and so

$$x_1 + x'_2 = x_2 + x'_1,$$

i.e.,

$$g_1 = g_2.$$

It remains to prove (iii). Let

$$x_i, x'_i \in X, \quad x_i - x'_i = g \quad (i = 1, 2, 3).$$

Then it follows from

$$x_1 - x'_1 \preceq^* x_2 - x'_2 \preceq^* x_3 - x'_3$$

that

$$x_1 + x'_2 \preceq x_2 + x'_1, x_2 + x'_3 \preceq x_3 + x'_2,$$

and according to the proof of item (5),

$$x_1 + x'_2 + x'_3 \preceq x_2 + x'_1 + x'_3, \quad x_2 + x'_3 + x'_1 \preceq x_3 + x'_2 + x'_1,$$

whence by virtue of the transitivity of the relation  $\preceq$ , it follows that

$$x_1 + x'_2 + x'_3 \preceq x_3 + x'_2 + x'_1.$$

It remains to apply (5.6).

(8) We have obtained that  $(G, +, \preceq^*)$  is a linearly ordered abelian group. Later on, to simplify notation, we shall write  $\preceq$  instead of  $\preceq^*$ , and  $\underline{a} \succ \underline{b}$  instead of  $\neg(a \preceq b)$ . We shall show that this is an Archimedean group, i.e. (see [51]), that for any  $g_1, g_2 > 0$  one can find  $n \in \mathbb{N}$  such that  $ng_1 > g_2$ . Let

$$g_1 = \beta^0(\underline{a}_1) - \beta^0(\underline{a}'_1), \quad g_2 = \beta^0(\underline{a}_2) - \beta^0(\underline{a}'_2).$$

Then  $g_1 > 0$ , i.e.,  $\beta^0(\underline{a}_1) > \beta^0(\underline{a}'_1)$ , follows  $\underline{a}'\beta'\underline{a}_1$ , and by virtue of property (5), we can find  $n \in \mathbb{N}$  such that

$$\underbrace{\underline{a}'_1, \underline{a}'_1, \dots, \underline{a}'_1}_n, \underline{a}_2 \beta' \underbrace{\underline{a}_1, \underline{a}_1, \dots, \underline{a}_1}_n.$$

But then

$$n\beta^0(\underline{a}'_1) + \beta^0(\underline{a}_2) < n\beta^0(\underline{a}_1) + \beta^0(\underline{a}'_2),$$

and so  $ng_1 \succ g_2$ , which is what had to be proved.

(9) We now make use of Hölder's theorem [51, Chapter VI, 3, paragraph 4], according to which any Archimedean group is isomorphic to some subgroup of an additive group of real numbers with its natural ordering. Denoting this isomorphism by  $\mathcal{L}(\mathcal{L} : G \rightarrow \mathbb{R})$ , we prove that the function

$$L : A \rightarrow \underline{R}, L(a) = \mathcal{L}(\beta^0(a))$$

is the function we are looking for, i.e.,

$$\underline{a}\beta\underline{b} \Leftrightarrow \sum_{i=1}^{n(\underline{a})} L(a_i) \leq \sum_{j=1}^{n(\underline{b})} L(b_j) \quad (5.8)$$

$$\forall a_i, b_j \in A, \quad n_1, n_2 \in \mathbb{N}, \quad \underline{a} = (a_i)_{1 \leq i \leq n_1}, \quad \underline{b} = (b_j)_{1 \leq j \leq n_2}.$$

For this purpose we must prove that

$$\mathcal{L}(\beta^0(\underline{a})) = \sum_{i=1}^{n_1} \mathcal{L}(\beta^0(a_i)), \quad (5.9)$$

or, what is equivalent,

$$\mathcal{L}(\beta^0(\underline{b})) = \sum_{j=1}^{n_2} \mathcal{L}(\beta^0(b_j)). \quad (5.10)$$

We shall prove the relation (5.9) by induction. Let  $n_1 = 2$ . Then  $\underline{a} = \overline{a_1 a_2}$ , and so

$$\beta^0(\underline{a}) = \beta^0(\overline{a_1 a_2}) = \beta^0(a_1) + \beta^0(a_2).$$

If we suppose now that the required assertion is true for  $n_1 = n$ , we obtain

$$\underline{a} = (a_1, a_2, \dots, a_{n+1}) = \overline{\underline{a}^n, a_{n+1}},$$

where  $\underline{a}_n = (a_1, a_2, \dots, a_n)$  and

$$\beta^0(\underline{a}) = \beta^0(\underline{a}^n) + \beta^0(a_{n+1}) = \sum_{i=1}^{n+1} \beta^0(a_{n+1}).$$

It remains to make use of the fact that  $\mathcal{L}$  is an isomorphism, and thus the relations (5.9) and (5.10) are valid along with the relation (5.8).

The reverse statement, i.e., that if  $L : A \rightarrow \mathbb{R}$  and (5.8) holds then  $\beta$  is a statistical preference on  $A$ , is checked directly.

(10) To prove the single-valuedness of the form of the function  $L$ , suppose that  $L_1 : A \rightarrow \mathbb{R}$ , and a relation similar to (5.8) is valid on substituting  $L$  by  $L_1$ .

Obviously, this implies that

$$\underline{a}\beta^0\underline{b} \Leftrightarrow \sum_{i=1}^{n(\underline{a})} L_1(a_i) \leq \sum_{j=1}^{n(\underline{b})} L_1(b_j),$$

and therefore we have the right to introduce the function

$$\overline{\mathcal{L}}_1 : X \rightarrow \mathbb{R}, \quad \overline{\mathcal{L}}_1(\beta^0(\underline{a})) = \sum_{i=1}^{n(\underline{a})} L_1(a_i).$$

For this function the relations

$$\begin{aligned}\overline{\mathcal{L}}_1(x+y) &= \overline{\mathcal{L}}_1(x) + \overline{\mathcal{L}}_1(y), \quad \forall x, y \in X, \\ \overline{\mathcal{L}}_1(x) \leq \overline{\mathcal{L}}_1(y) &\Leftrightarrow x \preceq y, \quad \forall x, y \in X,\end{aligned}$$

are valid, and so if

$$x - y = x'_1 - y'_1$$

in the sense of the group  $G$ , then from

$$x + y' = x' + y \in X$$

it follows that

$$\overline{\mathcal{L}}_1(x) + \overline{\mathcal{L}}_1(y') = \overline{\mathcal{L}}_1(x') + \overline{\mathcal{L}}_1(y),$$

i.e.,

$$\overline{\mathcal{L}}_1(x) - \overline{\mathcal{L}}_1(y) = \overline{\mathcal{L}}_1(x') - \overline{\mathcal{L}}_1(y').$$

Analogously, we are justified in introducing the function

$$\mathcal{L}_1 : G \rightarrow \mathbb{R}, \quad \mathcal{L}_1(g) = \overline{\mathcal{L}}_1(x) - \overline{\mathcal{L}}_1(x'),$$

where  $g = x - x'$ . And at the same time,

$$\mathcal{L}_1(g_1 + g_2) = \mathcal{L}_1(g_1) + \mathcal{L}_1(g_2), \quad \mathcal{L}_1(g_1) \leq \mathcal{L}_1(g_2) \Leftrightarrow g_1 \preceq^* g_2,$$

i.e.,  $\mathcal{L}_1$  is an isomorphic mapping of the Archimedean group  $(G, +, \preceq^*)$  into the additive subgroup  $\mathcal{L}_1(G)$  of real numbers. Thus,  $\mathcal{L}_1(G)$  is isomorphic to  $G$  and  $G$  is isomorphic to  $\mathcal{L}(G)$ , i.e.,  $\mathcal{L}_1(G)$  and  $\mathcal{L}(G)$  are isomorphic.

We now make use of a lemma from [51, Chapter VI] that says that if  $A$  and  $B$  are subgroups of an additive group of real numbers with its natural ordering, and  $\varphi$  is a monotonic homomorphism of  $A$  onto  $B$ , then there exists a real number  $r \geq 0$  such that for all  $a \in A$ , the relation  $a\varphi = ar$  is satisfied. In our case we have not merely a homomorphism, but an isomorphism, and therefore  $r > 0$ , and we obtain that

$$\mathcal{L}_1(g) = r\mathcal{L}(g).$$

In the particular case

$$g = a \in A,$$

this implies that

$$L_1(a) = rL(a) \quad \text{for all } a \in A.$$

The theorem is proved.

### 5.3 Statistical Regularity and Criterion Choice

Let the ordered triple  $Z = (\Theta, U, L)$  be a decision scheme in which  $\Theta$  and  $U$  are arbitrary nonempty sets, and  $L : \Theta \times U \rightarrow \mathbb{R}$  is a real function satisfying the following two conditions:

- (1)  $\inf\{L(\theta, u); \theta \in \Theta, u \in U\} > -\infty$ ;
- (2)  $\sup\{L(\theta, u); \theta \in \Theta\} < +\infty, \quad \forall u \in U.$

We denote the class of all decision schemes by  $\mathbb{Z}$ , and the subclass of those for which the set  $\Theta$  is fixed by  $\mathbb{Z}(\Theta)$ . The number  $L(\theta, u)$  will be interpreted as losses from making the decision  $u$  when the value of the unknown parameter equals  $\theta$ . Our goal here is to understand how, on the basis of what reasoning, to every decision scheme  $Z \in \mathbb{Z}$  one can associate to the function  $L_Z^* : U \rightarrow \mathbb{R}$ —the “criterion,” interpreted as  $L_Z^*(u)$ —an a priori estimate of the result of the decision  $u$ .

Let the set  $\Theta$  be fixed. Denote, as in Chapter 4, the set of all finite additive probabilities on  $\Theta$  by  $PF(\Theta)$ , i.e.,

$$PF(\Theta) = \left\{ p \in (2^\Theta \rightarrow [0, 1]) : p(\Theta) = 1, p(A \cap B) = p(A) + p(B \setminus A) \quad \forall A, B \subseteq \Theta \right\},$$

and let  $\mathbf{P}(\Theta)$  be the set of all statistical regularities on  $\Theta$ , and  $M(\Theta)$  the Banach space of all bounded real functions on  $\Theta$  with norm

$$\|f\| = \sup_{\theta \in \Theta} \|f(\theta)\|.$$

Recall Definition 5.1 and consider the mapping  $\kappa : \mathbf{P}(\Theta) \rightarrow \Pi(\Theta)$ , defined in the following way. If

$$P \in \mathbf{P}(\Theta), \quad \pi = \kappa(P), \quad Z = (\Theta, U, L) \in \mathbb{Z}(\Theta), \quad \pi(Z) = L_Z^*(\cdot),$$

then

$$L_Z^*(u) = \sup_{p \in P} \int L(\theta, u) p(d\theta) \quad \text{for all } u \in U.$$

We have the following theorem.

**Theorem 5.2.**  $\kappa_{\mathbb{Z}(\Theta)}(\mathbf{P}(\Theta)) = \Pi_1(\mathbb{Z}(\Theta)).$

*Proof.* Let  $\pi \in \Pi_1(\Theta)$ . Define on  $M(\Theta)$  a functional  $\varphi = \varphi_\pi(\cdot)$  of the following form. To any function  $f \in M(\Theta)$  associate the triple

$$Z_f = (\Theta, U, L_f) \in \mathbb{Z}(\Theta),$$

where  $U = \{u\}$  is a one-point set, and the function  $L_f$  is defined with the help of the function  $f$  in the following way:

$$L_f(\theta, u) = f(\theta) \quad \forall \theta \in \Theta.$$

Since  $Z_f \in \mathbb{Z}(\Theta)$ , we, knowing the criterion choice rule  $\pi$ , can find the “function”  $L_{Z_f}^* = \pi(Z)$ . Its domain of definition  $U$  contains the unique element  $u$ . As the value of the functional  $\varphi(f)$  we take the number  $L_{Z_f}^*(u)$ , the value of the function  $L_{Z_f}^*$  on this element.

This definition makes sense, since the value  $\varphi(f)$  does not depend on the concrete form of the one-point set  $U$ .

**Lemma 5.1.** *The functional  $\varphi = \varphi_\pi$  possesses the following properties:*

**Condition 1** *If  $f_1, f_2 \in M(\Theta)$ ,  $f_1(\theta) \leq f_2(\theta)$  for all  $\theta \in \Theta$ , then  $\varphi(f_1) \leq \varphi(f_2)$ .*

**Condition 2** *If  $f_1, f_2 \in M(\Theta)$ ,  $a, b \in \mathbb{R}$ ,  $a \geq 0$ ,  $f_1(\theta) = af_2(\theta) + b$  for all  $\theta$ , then  $\varphi(f_1) = a\varphi(f_2) + b$ .*

**Condition 3**  *$\varphi(f_1 + f_2) \leq \varphi(f_1) + \varphi(f_2)$  for all  $f_1, f_2 \in M(\Theta)$ .*

*Proof.* To prove Conditions 1 and 2, consider a two-point set  $U = \{u_1, u_2\}$  and set

$$\begin{aligned} U_i &= \{u_i\} \quad (i = 1, 2), \\ L_i : \Theta \times U_i &\rightarrow \mathbb{R}, \quad L_i(\theta, u_i) = f_i(\theta) \quad (i = 1, 2), \\ L : \Theta \times U &\rightarrow \mathbb{R}, \quad L(\theta, u_i) = f_i(\theta) \quad (i = 1, 2), \\ Z_i &= (\Theta, U_i, L_i) \quad (i = 1, 2), \\ Z &= (\Theta, U, L). \end{aligned}$$

According to Condition 1 of Definition 5.1,

$$L_{Z_i}^*(u_i) = L_Z^*(u_i) = \varphi_\pi(f_i) \quad (i = 1, 2).$$

Therefore from  $f_1(\theta) \leq f_2(\theta)$ , for all  $\theta \in \Theta$ , with the help of Condition 3, it follows that

$$\varphi_\pi(f_1) \leq \varphi_\pi(f_2).$$

Condition (1) of Lemma 5.1 is proved.

Similarly, from  $a, b \in \mathbb{R}$ ,  $a \geq 0$ ,  $f_1(\theta) = af_2(\theta) + b$  for all  $\theta \in \Theta$ , it follows that

$$L(\theta, u_1) = aL(\theta, u_2) + b \quad \text{at all } \theta \in \Theta,$$

and in accordance with Condition 2,

$$L_Z^*(u_1) = aL_Z^*(u_2) + b,$$

or equivalently,

$$\varphi(f_1) = a\varphi(f_2) + b,$$

i.e., Condition 2 of Lemma 5.1 is also satisfied.

To prove Condition 3 of Lemma 5.1, denote by  $U'$  a three-point set  $U' = \{u_1, u_2, u_3\}$  and assume that

$$L'(\theta, u_i) = \begin{cases} f_i(\theta), & \text{if } i = 1, 2, \\ \frac{f_1(\theta) + f_2(\theta)}{2}, & \text{if } i = 3, \end{cases}$$

$$\begin{aligned} Z' &= (\Theta, U', L'), \\ L_3(\theta, u_3) &= L'(\theta, u_3) \\ Z_i &= (\Theta, \{u_i\}, L_i) \quad (i = 1, 2, 3). \end{aligned}$$

According to Condition 1, we have

$$L_{Z'}^*(u_i) = L_{Z_i}^*(u_i),$$

and since

$$L'(\theta, u) + L'(\theta, u_2) = 2L'(\theta, u_3),$$

it follows from Condition 3 that

$$L_{Z'}^*(u_1) + L_{Z'}^*(u_2) \geq 2L_{Z'}^*(u_3),$$

i.e.,

$$\varphi(f_1) + \varphi(f_2) \geq 2\varphi\left(\frac{f_1 + f_2}{2}\right) = \varphi(f_1 + f_2).$$

The lemma is proved.

Denote by  $G$  the set of all limited linear functionals  $g$  on  $M(\Theta)$  satisfying the following two conditions:

- (1)  $g(\mathbf{1}_\Theta) = 1$ ;
- (2)  $g(f) \geq 0$  if  $f(\theta) \geq 0 \forall \theta \in \Theta$ .

Obviously, if  $p \in PF(\Theta)$  and

$$g : M(\Theta) \rightarrow \mathbb{R}, \quad g(f) = \int f(\theta)p(d\theta), \quad (5.11)$$

then  $g \in G$ . We now prove the converse, namely that for any  $g \in G$  there exists  $p \in PF(\Theta)$  for which (5.11) holds. To this end, assume  $p(A) = g(\mathbf{1}_A)$  for all  $A \subseteq \Theta$ . Obviously we have  $p(\Theta) = 1$ ,  $p(A) \geq 0$ , and  $p(A \cup B) = p(A) + p(B \setminus A)$  for all  $A, B \subseteq \Theta$ . Thus  $p \in PF(\Theta)$ . It remains to prove (5.11).

We set an arbitrary  $f \in M(\Theta)$  and  $\varepsilon \in (0, 1)$  and construct according to them the decomposition  $E = \{e; i \in \overline{1, n}\}$  of the set  $\Theta$  into a limited number of disjoint subsets satisfying the condition

$$\sup f(e_i) - \inf f(e_i) < \varepsilon \quad \forall i \in \overline{1, n}.$$

Let

$$\bar{f}_i = \sup f(e_i) \quad i \in \overline{1, n}, \quad \bar{f} = \sum_{i=1}^n f_i \mathbf{1}_{e_i}.$$

Then

$$0 \leq \bar{f} - f \leq \varepsilon,$$

whence it follows that

$$0 \leq \varepsilon \mathbf{1}_\Theta - (\bar{f} - f) \leq \varepsilon,$$

and hence

$$0 \leq g(\bar{f} - f) = g(\varepsilon \mathbf{1}_\Theta) - g(\varepsilon \mathbf{1}_\Theta - \bar{f} + f) = \varepsilon - g(\varepsilon \mathbf{1}_\Theta - \bar{f} + f) \leq \varepsilon.$$

But

$$g(\bar{f}) = \sum_{i=1}^n g(\bar{f}_i \mathbf{1}_{e_i}) = \sum_{i=1}^n f_i p(e_i),$$

and at the same time,

$$0 \leq \int \bar{f}(\theta)p(d\theta) - \int f(\theta)p(d\theta) \leq \varepsilon.$$

It is similarly proved that

$$|g(f) - \int f(\theta)p(d\theta)| \leq 2\varepsilon.$$

Due to the arbitrariness of  $f$  and  $\varepsilon$ , the representation (5.11) is valid.

**Lemma 5.2.** *In order that the functional*

$$\varphi : M(\Theta) \rightarrow \mathbb{R},$$

satisfy Conditions 1–3, it is necessary and sufficient that it be representable in the form

$$\varphi(f) = \sup_{g \in G_\varphi} g(f), \quad \forall f \in M(\Theta),$$

where  $G_\varphi$  is some nonempty subset of the set  $G$ .

*Proof.* Necessity. Choosing an arbitrary function  $f_0 \in M(\Theta)$ , denote by  $M_0$  a two-dimensional linear subspace of the space  $M(\Theta)$ , generated by functions  $f_0$  and  $\mathbf{1}_\Theta$ :

$$M_0 = \{f \in M(\Theta) : \exists a, b \in \mathbb{R}, f = af_0 + b\mathbf{1}_\Theta\}.$$

Define further on the set  $M_0$  a bounded linear functional

$$g_0 : M_0 \rightarrow \mathbb{R}, \quad g_0(af_0 + b) = a\varphi(f_0) + b.$$

It is easy to see that  $g_0(f) \leq \varphi(f)$  for all  $f \in M_0$ . Indeed, if

$$a \geq 0, \quad f = af_0 + b\mathbf{1}_\Theta,$$

then

$$g_0(f) = a\varphi(f_0) + b = \varphi(f).$$

If  $a < 0$ , note that from

$$\varphi(f) + \varphi(-f) \geq 2\varphi(f - f) = 2\varphi(0 \cdot \mathbf{1}_\Theta) = 0$$

it follows that  $\varphi(-f) \geq -\varphi(f)$ .

Hence when

$$f = af_0 + b\mathbf{1}_\Theta,$$

then

$$\varphi(f) \geq -\varphi(-f) = -[(-a)\varphi(f_0) - b] = a\varphi(f_0) + b = g_0(f).$$

So, due to Conditions 2 and 3,  $\varphi$  is a gauge function on  $M(\Theta)$ , and  $g_0(f) \leq \varphi(f)$  for  $f \in M_0$ . Then by the Hahn–Banach theorem (as in [15]), there exists a linear functional  $g_1$  on  $M(\Theta)$  such that

$$g_1(f) = g_0(f), \quad f \in M_0(\Theta),$$

and

$$g_1(f) \leq \varphi(f), \quad \forall f \in M(\Theta).$$

We show that  $g_1 \in G$ . Since the functional  $g_1$  is bounded and linear, it remains only to check that

$$g_1(\mathbf{1}_\Theta) = 1 \quad \text{and} \quad g_1(f) \geq 0$$

when  $f \geq 0$ .

Since  $\mathbf{1}_\Theta \in M_0$ , the first of these relations is obvious, and the second follows from the following argument.

If  $f \geq 0$ , then  $-f \leq 0$ , and thus according to Condition 1,

$$\varphi(-f) \leq \varphi(0 \cdot \mathbf{1}_\Theta) = 0.$$

Therefore

$$g_1(f) = -g(-f) \geq -\varphi(-f) \geq 0.$$

So  $g_1 \in G$ . And what is more,

$$g_1 \in \{g \in G : g(f) \leq \varphi(f) \quad \forall f \in M(\Theta)\} \stackrel{\text{def}}{=} G_\varphi.$$

It remains only to note that on the one hand,

$$\varphi(f) \geq \sup_{g \in G_\varphi} g(f),$$

since

$$\varphi(f) \geq g(f) \quad \text{for all} \quad g \in G_\varphi, \quad f \in M(\Theta),$$

and on the other hand,

$$\varphi(f) \leq \sup_{g \in G_\varphi} g(f),$$

since for any function  $f \in M(\Theta)$ , denoting it by  $f_0$ , one can construct, as it was done above, the functional  $g_1 \in G_\varphi$  such that

$$\varphi(f_0) = g_1(f_0).$$

The necessity is proved.

Sufficiency. Let  $G_0 \subseteq G$ ,

$$\varphi : M(\Theta) \rightarrow \mathbb{R}, \quad \varphi(f) = \sup_{g \in G_0} g(f).$$

We prove Condition 1. Let  $f_1, f_2 \in M(\Theta)$ ,

$$f_1(\theta) \leq f_2(\theta) \quad \text{for all} \quad \theta \in \Theta.$$

Then  $g(f_2 - f_1) \geq 0$  for all  $g \in G$ , and so due to the linearity of all  $g \in G$ ,

$$\varphi(f_1) = \sup_{g \in G_0} g(f_1) \leq \sup_{g \in G_0} g(f_2) = \varphi(f_2).$$

Condition 2 is obvious. Condition 3 follows from the fact that

$$\begin{aligned} \varphi(f_1 + f_2) &= \sup_{g \in G_0} g(f_1 + f_2) = \sup_{g \in G_0} [g(f_1) + g(f_2)] \\ &\leq \sup_{g \in G_0} g(f_1) + \sup_{g \in G_0} g(f_2) = \varphi(f_1) + \varphi(f_2). \end{aligned}$$

Lemma 5.2 is now entirely proved.

To complete the proof of Theorem 5.2, we associate to every  $\pi \in \Pi_1(\Theta)$  a functional  $\varphi = \varphi_\pi$  as is done in Lemma 5.1, find the appropriate subset  $G_\varphi \subseteq G$  from Lemma 5.2, and then, associating for every  $g \in G_\varphi$  the corresponding (in the sense of (5.11)) distribution  $p = p_g \in PF(\Theta)$ , we set  $P_\varphi = \{p_g, g \in G_\varphi\}$ . Obviously  $P_\varphi = [P_\varphi]$  in the sense of the topology  $\tau(\Theta)$ .

We obtain

$$\sup_{g \in G_\varphi} g(f) = \sup_{p \in P_\varphi} \int f(\theta) p(d\theta) = \max_{p \in P_0} \int f(\theta) p(d\theta)$$

for all  $f \in M(\Theta)$  by the definition of completion in the topology  $\tau(\Theta)$ .

It is proved similarly that for any decision scheme  $Z = (\Theta, U, L)$ ,  $u \in U$ , and  $\pi \in \Pi_1(\Theta)$ , for  $L_Z^* = \pi(Z)$  we have the equality

$$L_Z^*(u) = \varphi(L(\cdot, u)) = \max_{p \in P_\varphi} \int L(\theta, u) p(d\theta).$$

Therefore  $\Pi_1(\Theta) \subseteq \kappa(\mathbf{P}(\Theta))$ .

To prove the reverse inclusion we note that when  $P \in \mathbf{P}(\Theta)$ , one can construct for any  $p \in P$  the functional  $g = g_p$  defined by formula (5.11), and then, setting  $G_0 = \{g_p, p \in P\}$ , we check that according to Lemma 5.2, the functional

$$\varphi : M(\Theta) \rightarrow \mathbb{R}, \quad \varphi(f) = \sup_{g \in G_0} g(f)$$

satisfies Conditions 1–3. Similarly, assuming for any decision scheme  $Z = (\Theta, U, L)$  the criterion  $L_Z^*$  to be equal to

$$L_Z^* : U \rightarrow R, \quad L_Z^*(u) = \varphi(L(\cdot, u)),$$

we arrive at the criterion choice rule  $\pi : Z \rightarrow L_Z^*$  from the class  $\Pi_1(\Theta)$ .

The theorem is proved.



# Chapter 6

## Experiment in Decision Problems

As I have already noticed many times, experiment means nothing at all unless it is interpreted by a theory.

---

Max Born

### 6.1 Preliminaries

So far, we have considered the case in which we know beforehand the set  $\Theta$  of values of the unknown parameter  $\theta$  and the function  $L$ , that is, the scheme  $Z = (\Theta, U, L)$ . It follows from Theorem 5.2 that for a univalent assignment of the criterion  $L_Z^*$ , it is sufficient to know as well the statistical regularity  $P$  on  $\Theta$  describing the supposed behavior of  $\theta \in \Theta$ . In this way, the pair  $S = (Z, P)$  becomes a complete description of the decision problem  $T = \{S, L_Z^*\}$  (Remark 5.1).

However, we may ask the following question. If the parameter  $\theta$  is not observable, is it obligatory to introduce  $\theta$  into consideration? To answer this question we have to return to notation introduced in Chapter 2. Namely, the quadruple  $S_m = (\Theta, U, L, P)$  was called the matrix model, while the triple  $S_l = (U, \psi, Q)$ , where  $\psi : U \rightarrow 2^{\mathbb{R}}$ , was called the lottery model of decision-making situations.<sup>1</sup> Is it not simpler in this case instead of the matrix model  $S_m$  to consider the lottery model  $S_l$ , where  $Q = \{Q_u; u \in U\}$  is the family of regularities on  $\mathbb{R}$  describing the behavior of losses for different  $u \in U$ ?

We shall show that the matrix model  $S_m$  is equivalent to the lottery model  $S_l$  in the case of a finite set  $U$  and independent consequences. For the situations with stochastic regularity this equivalence was shown in Chapter 2; see also [42]. Here we show that this equivalence takes place for statistical regularity as well.

It is easy to see that in order to pass from  $S_m$  to  $S_l$  it is sufficient to construct the statistical regularity in the following form:

---

<sup>1</sup> Recall that in this book we consider only decision situations with  $C = \mathbb{R}$ .

$$Q_u = \{q \in PF(\mathbb{R}) : \exists p \in P, \forall A \subseteq \mathbb{R}, \quad q(A) = p(\{\theta \in \Theta : L(\theta, u) \in A\})\}. \quad (6.1)$$

In order to pass from  $S_l$  to  $S_m$ , one can construct the set  $\Theta$  and statistical regularity  $P_\Theta$  as follows:

$$\begin{aligned} \Theta &= \{(r_1, r_2, \dots, r_n) : r_j \in \mathbb{R}_j, j = (\overline{1, n})\} \\ L((r_1, r_2, \dots, r_n), u_j) &\stackrel{\text{def}}{=} r_j, \quad j = (\overline{1, n}), \\ P_\Theta &\stackrel{\text{def}}{=} \left\{ p \in PF(\mathbb{R}^n) : \exists q_1 \in Q_{u_1}, q_2 \in Q_{u_2}, \dots, q_n \in Q_{u_n}, \right. \\ &\quad \forall A_1, A_2, \dots, A_n, A_j \subseteq \mathbb{R}, \\ &\quad \left. j = (\overline{1, n}), p(A_1, A_2, \dots, A_n) = \prod_{j=1}^n q_j(A_j) \right\}. \end{aligned}$$

It is clear that for the aims of decision-making, the lottery model does as well as the matrix model. However, the situation changes sharply when before making a decision we want to conduct an experiment to obtain some additional information about the decision situation, i.e., to take advantage of observation [50, 57, 60].

In Chapter 2 we have already mentioned that two types of experiment are possible in decision systems: the experiment with observation of consequences of decisions and the experiment with observation of the unknown parameter. However, experiments of the first type, observation of consequences, are feasible in parametric as well as in nonparametric situations, whereas experiments of the second type, observation of the unknown parameter, are feasible only in parametric situations.

In a matrix decision problem  $T_m = \{S_m, L_{Z_m}^*\}$  (or in view of Remark 5.1, just  $S_m$ ), the observation is some function  $h_m$  on  $\Theta$ , i.e.,  $h_m : \Theta \rightarrow W_m$ , where  $W_m$  is the set of all possible values of the observation  $h_m$ .<sup>2</sup> Therefore the decision situation model  $S_m$  with regard to the experiment  $h_m$  can be formalized thus:

$$S_m^{h_m} = (\Theta, d_m, \mathcal{L}, P),$$

where

$$d_m : W_m \rightarrow U, \quad d_m \in D_m = \{W_m \rightarrow U\},$$

and where  $D_m$  is the set of all possible “experiment utilization strategies” or *decision functions* [14], and

<sup>2</sup> Notice that in refusing to consider here random functions of  $\theta$ , we do not restrict in any way the generality of our reasoning, because the case of a random function of  $\theta$  can be reduced to the one under consideration by means of an obvious change of notation (the direct product  $\Theta \times \Omega$ , where  $\Omega$  is a set of elementary events, is assumed to be the new set of values of the unknown parameter  $\theta$ ).

$$\mathcal{L} : \Theta \times D_m \rightarrow \mathbb{R}, \quad \mathcal{L}(\theta, d_m) = L(\theta, d_m(h_m(\theta)))$$

is the loss function recalculated in the natural way. Thus we distinguish the matrix decision problems with and without experiment as

$$T_m^{h_m} = \{S_m^{h_m}, L_{Z_m}^*\}$$

and

$$T_m = \{S_m, L_{Z_m}^*\},$$

or taking into account Remark 5.1, simply  $S_m^{h_m}$  and  $S_m$ .

In the lottery decision problem  $T_l = \{S_l, L_{Z_l}^*\}$ , taking into account (5.3) and (6.1), the criterion  $L_{Z_l}^*$  has the form

$$L_{Z_l}^*(u) = \max_{q_u^*} \max_{q_u \in Q_u} \int L(r) q_u(dr), \quad (6.2)$$

where

$$q_u^* = \arg \max_{q_u \in Q_u} \int L(r) q_u(dr), \quad (6.3)$$

and where  $L(r)$  is the loss function defined on consequences  $r \in \mathbb{R}$ .

Therefore we can consider the lottery models with and without experiment

$$T_l^{h_l} = \{S_l^{h_l}, L_{Z_l}^*\}, \quad T_l = \{S_l, L_{Z_l}^*\},$$

where

$$S_l^{h_l} = (d_l, \psi(\cdot), Q), \quad \psi : U \rightarrow 2^{\mathbb{R}}, \quad d_l \in D_l = \{W_l \rightarrow U\},$$

and where the observation  $h_l$  is some function  $h_l : \mathbb{R} \rightarrow W_l$ , where  $W_l$  is the set of all possible values of the observation  $h_l$ .

In both cases, matrix and lottery, it is quite natural to present the gain from the observation as the difference between the “best average results” in  $S$  and  $S^h$  [62, 77]. If we used a lottery model, then the dependence of the consequence of the decision  $u \in U$  on the observation  $w \in W$  would have to be described by a family of joint statistical regularities on  $\mathbb{R} \times W$ . The formulas and reasoning that await us on this path are extremely complex and provide nothing new in comparison to the corresponding formulas for the matrix model. Therefore, in what follows, unless otherwise stipulated, we use only the matrix model with experiment of the second type.

We say that the experiment  $h_1$  is *finer* than  $h_2$ , and  $h_2$  is *coarser* than  $h_1$ , if from the equality

$$h_1(\theta_1) = h_1(\theta_2)$$

follows the equality

$$h_2(\theta_1) = h_2(\theta_2)$$

for any  $\theta_1, \theta_2 \in \Theta$ . We denote by  $H(\Theta)$  the class of all experiments on  $\Theta$  and introduce notation for its subclasses:

$$\begin{aligned} H_*(\Theta) &= \{h \in H(\Theta) : h(\theta_1) = h(\theta_2), \forall \theta_1, \theta_2 \in \Theta\}, \\ H^*(\Theta) &= \{h \in H(\Theta) : h(\theta_1) \neq h(\theta_2), \text{ if } \theta_1 \neq \theta_2\}. \end{aligned}$$

Obviously the observations from the class  $H_*(\Theta)$  are the coarsest, and the observations from the class  $H^*(\Theta)$  are the finest.

Since the set  $D$  of all possible strategies contains all constants, obviously the risk satisfies  $\rho(S^h) \leq \rho(S)$ , whatever the experiment  $h$  (see the definition of risk of a decision problem in Chapter 5). Furthermore, if  $h \in H_*(\Theta)$ , then

$$\rho(S^h) = \rho(S),$$

and if  $h_1$  is finer than  $h_2$ , then

$$\rho(S^{h_1}) \leq \rho(S^{h_2}).$$

Therefore it is quite natural to call the difference  $\rho(S) - \rho(S^h)$  the *informativity of the experiment (observation)  $h$  for the decision problem  $S$* .<sup>3</sup> We shall denote it by  $\text{INF}(h/S)$ .

It is easy to prove that if  $h_* \in H_*(\Theta)$ ,  $h \in H(\Theta)$ , and  $h^* \in H^*(\Theta)$ , then

$$0 = \text{INF}(h_*/S) \leq \text{INF}(h/S) \leq \text{INF}(h^*/S) = \max_{h' \in H(\Theta)} \text{INF}(h'/S).$$

We denote the last value by  $v(S)$  and call it the *measure of uncertainty*, or just the *uncertainty*, of the decision problem  $S$ . It is equal to

$$v(S) = \inf_{u \in U} \sup_{p \in \mathcal{P}} \int L(\theta, u) p(d\theta) - \sup_{p \in \mathcal{P}} \int [\inf_{u \in U} L(\theta, u)] p(d\theta)$$

and can—in another presentation—be used for the definition of the informativity of an experiment, since (Theorem 6.3)

$$\text{INF}(h/S) = v(S) - v(S^h).$$

<sup>3</sup> We quite deliberately use here the word “informativity” instead of “usefulness” or “value,” which may seem more appropriate at first glance. The reasons will be clear by the end of Chapter 7.

This brings to mind Shannon's definition of information quantity [75], and as we shall see later, the similarity here is rather deep.

We denote by  $\mathbf{S}(\Theta)$  the class of all decision problems of the form  $S = (\Theta, \cdot, \cdot, \cdot)$ . The following property of an experiment (Theorem 6.1) is of great importance for our study. The experiment  $h_1 \in H(\Theta)$  is finer than the experiment  $h_2 \in H(\Theta)$  if and only if

$$\text{INF}(h_1/S) \geq \text{INF}(h_2/S)$$

for all  $S \in \mathbf{S}(\Theta)$ .

In other words, if  $h_1$  is not finer than  $h_2$ , and  $h_2$  is not finer than  $h_1$ , one can find two decision problems  $S_1$  and  $S_2$  such that for one of them the informativity will be greater in the experiment  $h_1$ , while for the other, it will be greater in the experiment  $h_2$ .

It is clear, however, that informativity is not the only property that we use to evaluate an experiment in practice. A property such as "complexity," defined as the capacity of the set  $W = h(\Theta)$  of values of this experiment, is also important. Therefore, it seems natural to introduce the notion of an optimal experiment. The experiment  $h \in H(\Theta)$  is called *optimal for the decision problem*  $S \in \mathbf{S}(\Theta)$  if

$$\text{INF}(h/S) \geq \text{INF}(h_1/S)$$

for any other experiment  $h_1 \in H(\Theta)$ , and at the same time, if

$$\text{INF}(h_1/S) = \text{INF}(h/S),$$

then

$$\text{Card}(h(\Theta)) \leq \text{Card}(h_1(\Theta)).$$

Moreover, it turns out (Theorem 6.5) that for every decision problem one can find an optimal experiment, and every experiment is optimal for some decision problem. Moreover, given decision problem  $S = (\Theta, \cdot, \cdot, \cdot) \in \mathbf{S}(\Theta)$ , the experiment  $h \in H(\Theta)$  optimal for  $S$ , and the finite set  $U$ , then

$$\text{Card}(h(\Theta)) \leq \text{Card}(U).$$

Let  $h$  be an observation on  $\Theta$ . We call the function

$$\mathcal{K}_h : \Theta \rightarrow 2^\Theta, \quad \mathcal{K}_h(\theta) = \{\theta' \in \Theta : h(\theta') = h(\theta)\}$$

a *classification* on  $\Theta$  generated by the experiment  $h$ . Obviously,  $\mathcal{K}_h \in H(\Theta)$  for any  $h \in H(\Theta)$ . Note here that classifications on  $\Theta$  make up a subclass of the class of all experiments on  $\Theta$  that plays a special role:

$$\text{INF}(h_1/S) = \text{INF}(h_2/S) \quad \forall S \in \mathbf{S}(\Theta)$$

if and only if  $\mathcal{K}_{h_1} = \mathcal{K}_{h_2}$ . In other words, in order to find the informativity of some experiment, it is sufficient to know the informativity of the classification generated by this experiment.

Making use of this terminology, one can reformulate all that was said above about optimal experiments in the following way: for any decision problem one can find the optimal classification, and any classification is optimal for some decision problem [40].

## 6.2 Informativity of Experiments

As noted previously, the introduction of the set  $\Theta$  makes sense only as far as we plan to observe the value of the parameter  $\theta \in \Theta$  before making a decision. So we introduce the following definitions.

**Definition 6.1.** An observation (an experiment) on  $\Theta$  is any function (not necessarily real-valued) with  $\Theta$  as its domain of definition. The class of all observations on  $\Theta$  is denoted by  $H(\Theta)$ .

Note that a random observation is a particular case of this definition, because when the decision situation is formalized in the form of a decision problem we could choose the set  $\Theta$  so that all the observed variables are functions of  $\theta \in \Theta$ . For instance, if we observe a random function

$$\xi : \Theta \times \Omega \rightarrow W,$$

where  $(\Omega, A, P)$  is the probability space, then we may define  $\Theta'$  as the Cartesian product  $\Theta \times \Omega$ , and the “former” random function  $\xi$  becomes an observation on  $\Theta'$ .

**Definition 6.2.** We call the value

$$\text{INF}(h/S) = \rho(S) - \rho(S^h)$$

the *informativity* of the experiment  $h \in H(\Theta)$  relative to the decision problem

$$S = (\Theta, U, L, P) \in \mathbf{S}(\Theta).$$

where  $S^h$  is a decision problem defined as follows:

$$\begin{aligned}
d &: h(\Theta) \rightarrow U, \\
d &\in D = \{h(\Theta) \rightarrow U\}, \\
\mathcal{L} &: \Theta \times D \rightarrow \mathbb{R}, \\
\mathcal{L}(\theta, d) &= L(\theta, d(h(\theta))), \\
S^h &= (\Theta, d, \mathcal{L}, P),
\end{aligned}$$

where  $h(\Theta)$  is the image of the set  $\Theta$  under the mapping  $h$ .

*Remark 6.1.* Interpreting the risk  $\rho(S)$  as the optimal estimate of the guaranteed outcome that we have before the decision-making in decision problem  $S$ , it is natural to consider the difference

$$[\rho(S) - \rho(S^h)]$$

as the “value” (the utility) of the experiment  $h$  in the given decision problem  $S$ . In Chapter 7 we show that for one important class of decision problems any “reasonable” measure of information quantity contained in the experiment is proportional to this difference. Therefore we prefer to call it not the “value,” but the “informativity” of experiment, underlining only the feature which seems essential here [62].

Finally, we shall say that the experiment  $h_1 \in H(\Theta)$  is finer than the experiment  $h_2 \in H(\Theta)$  if for any  $\theta_1, \theta_2 \in \Theta$ , from  $h_1(\theta_1) = h_1(\theta_2)$  it follows that  $h_2(\theta_1) = h_2(\theta_2)$ .

**Theorem 6.1.** *The observation  $h_1 \in H(\Theta)$  is finer than the observation  $h_2 \in H(\Theta)$  if and only if*

$$\text{INF}(h_1/S) \geq \text{INF}(h_2/S)$$

for all  $S \in \mathbf{S}(\Theta)$ .

*Proof.* Let

$$W_i = h_i(\Theta), \quad D_i = \{W_i \rightarrow U\}, \quad S_i = S^{h_i} \quad (i = 1, 2),$$

and  $h_1$  is finer than  $h_2$ . Denote by  $h_1^{-1}(w)$  the complete preimage of the point  $w \in W_1$  under the mapping  $h_1$ . If  $\theta_1, \theta_2 \in h_1^{-1}(w)$ , then  $h_2(\theta_1) = h_2(\theta_2)$ , and we may introduce the function

$$f : W_1 \rightarrow W_2, \quad f(w) = h_2(h_1^{-1}(w)),$$

and then the function

$$F : D_2 \rightarrow D_1,$$

which is defined as follows: if  $d_2 \in D_2$ ,  $d_1 \in F(d_2)$ , then

$$d_1 : W_1 \rightarrow U, \quad d_1(w) = d_2(f(w)).$$

We clearly have  $d_1(h_1(\theta)) = d_2(h_2(\theta))$  for any  $\theta \in \Theta$ , and thus

$$L(\theta, d_1(h_1(\theta))) = L(\theta, d_2(h_2(\theta)))$$

for any  $d_2 \in D_2$ ,  $d_1 = F(d_2)$ . Hence we immediately obtain

$$\begin{aligned} \rho(S^{h_1}) &= \inf_{d_1 \in D_1} \max_{p \in P} \int_{\Theta} L(\theta, d_1(h_1(\theta))) p(d\theta) \\ &\leq \inf_{d_2 \in D_2} \max_{p \in P} \int_{\Theta} L(\theta, d_2(h_2(\theta))) p(d\theta) = \rho(S^{h_2}). \end{aligned}$$

To prove the converse, assume that  $h_1$  is not finer than  $h_2$ , i.e., there exist  $\theta_0, \theta_1 \in \Theta$  such that

$$h_1(\theta_0) = h_1(\theta_1)$$

and at the same time

$$h_2(\theta_0) \neq h_2(\theta_1).$$

Construct a decision problem of the following form:  $U = \{u_0, u_1\}$  is a two-point set,  $L : \Theta \times U \rightarrow \mathbb{R}$ ,

$$L(\theta, u) = \begin{cases} 1, & \text{if } \theta = \theta_i, \\ 0, & \text{if } \theta \neq \theta_i \end{cases} \quad (i = 0, 1).$$

It is easy to verify that for such decision problems the following relations are true:

$$\begin{aligned} \rho(S) &= \min_{u \in U} \max_{\theta \in \Theta} L(\theta, u) = 1, \\ \rho(S^{h_1}) &= \min_{d_1 \in D_1} \max_{\theta \in \Theta} L(\theta, d_1(h_1(\theta))) = 1 \end{aligned}$$

(since  $h_1(\theta_0) = h_1(\theta_1)$ , so that  $d_1(h_1(\theta_0)) = d_1(h_1(\theta_1))$ ,  $\forall d_1 \in D_1$ ), whence

$$\max_{\theta \in \Theta} L(\theta, d_1(h_1(\theta))) = 1, \quad \forall d_1 \in D_1.$$

On the other hand, taking  $d_2 \in D_2$  in the form

$$d_2(w) = \begin{cases} u_1, & \text{if } h_2(\theta_0) = w, \\ u_0, & \text{if } h_2(\theta_0) \neq w, \end{cases}$$

we obtain

$$L(\theta, d_2(h_2(\theta))) = 0$$

for all  $\theta \in \Theta$ , and so

$$\rho(S^{h_2}) = 0 < \rho(S^{h_1}) = 1.$$

The theorem is proved.

We define

$$\begin{aligned} H_* &= \{h \in H(\Theta) : h(\theta_1) = h(\theta_2), \forall \theta_1, \theta_2 \in \Theta\}, \\ H^* &= \{h \in H(\Theta) : h(\theta_1) \neq h(\theta_2), \text{ when } \theta_1 \neq \theta_2\}. \end{aligned}$$

**Theorem 6.2.** *Let*

$$h_* \in H_*, \quad h^* \in H^*, \quad h \in H(\Theta), \quad S \in \mathbf{S}(\Theta).$$

*Then*

$$0 = \text{INF}(h_*/S) \leq \text{INF}(h/S) \leq \text{INF}(h^*/S).$$

*Proof.* Obviously,  $h^*$  is finer than  $h$ , and  $h$  is finer than  $h_*$ ; therefore we have only to prove that

$$\text{INF}(h_*/S) = 0, \quad \text{i.e., } \rho(S^{h_*}) = \rho(S).$$

To this end, note that the set

$$D_* = h_*(\Theta)$$

consists of only one point, and hence

$$\begin{aligned} \rho(S^{h_*}) &= \min_{d_* \in D_*} \max_{p \in P} \int L(\theta, d_*(h_*(\theta))) p(d\theta) \\ &= \min_{u \in U} \max_{p \in P} \int L(\theta, u) p(d\theta) = \rho(S). \end{aligned}$$

The theorem is proved.

From Theorem 6.2 it follows that

$$\text{INF}(h^*/S) = \max_{h \in H(\Theta)} \text{INF}(h/S).$$

We denote this value by  $v(S)$  and call it *uncertainty of the decision problem S*.

**Theorem 6.3.** *For all*

$$S = (\Theta, U, L, P) \in \mathbf{S}(\Theta), \quad h \in H(\Theta),$$

we have the relations

$$\begin{aligned} v(S) &= \inf_{u \in U} \max_{p \in P} \int L(\theta, u) p(d\theta) - \max_{p \in P} \int [\inf_{u \in U} L(\theta, u)] p(d\theta), \\ \text{INF}(h/S) &= v(S) - v(S^h). \end{aligned}$$

*Proof.* We have to prove two equalities:

$$\rho(S^{h^*}) = \max_{p \in P} \int [\inf_{u \in U} L(\theta, u)] p(d\theta) \quad (6.4)$$

and

$$\max_{p \in P} \int [\inf_{u \in U} L(\theta, u) p(d\theta)] = \max_{p \in P} \int [\inf_{u \in U} L(\theta, d(h^*(\theta)))] p(d\theta), \quad (6.5)$$

where

$$D^* = (h^*(\Theta) \rightarrow U).$$

We rewrite the equality (6.4) as

$$\inf_{d \in D^*} \max_{p \in P} \int L(\theta, d(h^*(\theta))) p(d\theta) = \max_{p \in P} \int [\inf_{u \in U} L(\theta, u)] p(d\theta),$$

and then as the system of inequalities

$$\inf_{d \in D^*} \max_{p \in P} \int L(\theta, d(h^*(\theta))) p(d\theta) \leq \max_{p \in P} \int [\inf_{u \in U} L(\theta, u)] p(d\theta) \quad (6.6)$$

and

$$\inf_{d \in D^*} \max_{p \in P} \int L(\theta, d(h^*(\theta))) p(d\theta) \geq \max_{p \in P} \int [\inf_{u \in U} L(\theta, u)] p(d\theta). \quad (6.7)$$

Let  $\varepsilon > 0$ , and suppose that every  $\theta \in \Theta$  is associated with  $u_\theta \in U$  such that

$$L(\theta, u_\theta) \leq \inf_{u \in U} L(\theta, u) + \varepsilon \quad \text{for all } \theta \in \Theta.$$

Consider the strategy

$$d_\varepsilon : h^*(\Theta) \rightarrow U, d_\varepsilon(w) = u_\theta \quad \text{for } w = h^*(\theta).$$

For this strategy we have

$$\begin{aligned} \inf_{d \in D^*} \max_{p \in P} \int L(\theta, d(h^*(\theta))) p(d\theta) &\leq \max_{p \in P} \int L(\theta, d_\varepsilon(h^*(\theta))) p(d\theta) \\ &\leq \max_{p \in P} \int [\inf_{u \in U} L(\theta, u)] p(d\theta) + \varepsilon, \end{aligned}$$

which, by virtue of the arbitrariness of  $\varepsilon > 0$ , proves (6.6). It is straightforward to check (6.7):

$$\begin{aligned} \inf_{d \in D^*} \max_{p \in P} \int L(\theta, d(h^*(\theta))) p(d\theta) &\geq \inf_{d \in D^*} \max_{p \in P} \int [\inf_{u \in U} L(\theta, u)] p(d\theta) \\ &= \max_{p \in P} \int [\inf_{u \in U} L(\theta, u)] p(d\theta). \end{aligned}$$

This proves relation (6.4). To prove (6.5), it is sufficient to note that on the one hand,

$$\inf_{u \in U} L(\theta, u) \leq L(\theta, d(h(\theta)))$$

for all  $d \in D$ ,  $h \in H(\Theta)$ , and thus

$$\inf_{u \in U} L(\theta, u) \leq \inf_{d \in D^*} L(\theta, d(h^*(\theta))),$$

and on the other hand,

$$\inf_{d \in D^*} L(\theta, d(h^*(\theta))) \leq L(\theta, u) \quad \text{for all } u \in U.$$

Hence

$$\inf_{d \in D^*} L(\theta, d(h^*(\theta))) \leq \inf_{u \in U} L(\theta, u).$$

The theorem is proved.

It immediately follows from Theorem 6.3 that for the decision problem  $S = (\Theta, U, L, \{p\})$ , where  $\{p\}$  is a one-point set, we have

$$v(S) = \inf_{u \in U} \int L(\theta, u) p(d\theta) - \int [\inf_{u \in U} L(\theta, u)] p(d\theta),$$

and for the minimax decision problem  $S = (\Theta, U, L, PF(\Theta))$ , we have

$$v(S) = \inf_{u \in U} \sup_{\theta \in \Theta} L(\theta, u) - \sup_{\theta \in \Theta} \inf_{u \in U} L(\theta, u).$$

For the Bayesian decision problem,  $v(S)$  is given by following theorem.

**Theorem 6.4.** *Let  $(\Theta, \mathcal{A}, \mu)$  be a probability space,  $U$  an arbitrary set,  $L : \Theta \times U \rightarrow \mathbb{R}$  a bounded real function with the  $\mathcal{A}$ -measurable sections  $L(\cdot, u)$ ,  $u \in U$ ,*

and suppose the set  $P_\mu$  is defined by the formula

$$P_\mu = \{p \in PF(\Theta) : \forall A \in \mathcal{A}, p(A) = \mu(A)\}. \quad (6.8)$$

Then

$$\begin{aligned} \mathbf{S} &= (\theta, U, L, P_\mu) \in \mathbf{S}(\Theta), \\ L_S^*(u) &= \int L(\theta, u) \mu(d\theta) \quad \forall u \in U, \end{aligned} \quad (6.9)$$

$$\rho(S) = \inf_{u \in U} \int L(\theta, u) \mu(d\theta), \quad (6.10)$$

$$v(S) = \inf_{u \in U} \int L(\theta, u) \mu(d\theta) - \overline{\int} [\inf_{u \in U} L(\theta, u) \mu(d\theta)], \quad (6.11)$$

with  $\overline{\int}$  the upper integral of the (possibly unmeasurable) function  $\inf_{u \in U} L(\cdot, u)$ .

*Proof.* We have to prove the following relations:

**Relation 1**  $P_\mu \neq \emptyset$ .

**Relation 2**  $[P_\mu] = P_\mu$ .

**Relation 3**  $\int f(\theta) p(d\theta) = \int f(\theta) \mu d\theta$  for every  $\mathcal{A}$ -measurable function  $f \in M(\Theta)$  and every distribution  $p \in P_\mu$ .

**Relation 4**  $\max_{p \in P_\mu} \int f(\theta) p(d\theta) = \overline{\int} f(\theta) \mu(d\theta)$ .

Let us begin with the proof of Relation 1. Set

$$\varphi : M(\Theta) \rightarrow \mathbb{R}, \quad \varphi(f) = \overline{\int} f(\theta) \mu(d\theta),$$

and check that the functional introduced in this way satisfies Conditions 1–3 formulated in Lemma 5.1. Denote by  $F(f)$  the set of all measurable functions  $\bar{f} \in M(\Theta)$  such that  $\bar{f}(\theta) \geq f(\theta)$  for all  $\theta \in \Theta$ . According to the definition of upper integral,

$$\varphi(f) = \overline{\int} f(\theta) \mu(d\theta) = \inf_{\bar{f} \in F(f)} \int \bar{f}(\theta) \mu(d\theta).$$

Therefore Condition 1 of Lemma 5.1 follows directly from the inclusion  $F(f_2) \subseteq F(f_1)$ , which holds for all  $f_1 \leq f_2$ .

We now prove that Condition 2 holds. If  $f_1, f_2 \in M(\Theta)$ ,  $a, b \in \mathbb{R}$ ,  $a \geq 0$ ,

$$f_1 = af_2(\theta) + b \quad \forall \theta \in \Theta,$$

then associating to every function  $\bar{f} \in M(\Theta)$  the function

$$\bar{f}_2 = a\bar{f}_1 + b\mathbf{1}_\Theta,$$

we obtain that

$$\bar{f}_1 \in F(f_1) \iff \bar{f}_2 \in F(f_2)$$

and

$$\int \bar{f}_2(\theta)\mu(d\theta) = a \int \bar{f}_1(\theta)\mu(d\theta) + b,$$

which proves the required relationship. Condition 3 is obvious.

It remains to construct, as we did in the proof of Theorem 5.2, using the functional  $\varphi$ , the set

$$G_\varphi = \{g \in G : g(f) \leq \varphi(f) \forall f \in M(\Theta)\},$$

and then the set

$$P = \left\{ p \in PF(\Theta) : \exists g \in G_\varphi \forall f \in M(\Theta), g(f) = \int f(\theta)p(d\theta) \right\}.$$

And this is the desired set  $P_\mu$ . Indeed, if  $p \in P$ , then

$$\begin{aligned} p(A) &\leq \varphi(\mathbf{1}_A) = \mu(A), \quad \text{for all } A \in \mathcal{A}, \\ p(\Theta \setminus A) &\leq \varphi(\mathbf{1}_{\Theta \setminus A}) = \mu(\Theta \setminus A), \quad \text{for all } A \in \mathcal{A}, \end{aligned}$$

and thus

$$p(A) = \mu(A), \quad \text{i.e., } p \in P_\mu.$$

Conversely, let  $p \in P_\mu$ . We shall show that  $p \in P$ , i.e.,

$$\int f(\theta)p(d\theta) \leq \overline{\int f(\theta)\mu(d\theta)} \quad \text{for all } f \in M(\Theta).$$

To this end, taking an arbitrary  $\varepsilon > 0$ , choose the function  $\bar{f} \in F(\bar{f})$  such that

$$\int \bar{f}(\theta)\mu(d\theta) \leq \inf_{\bar{f}_1 \in F(f)} \int \bar{f}_1(\theta)\mu(d\theta) + \varepsilon.$$

Since this function is  $\mathcal{A}$ -measurable, there exists a partition  $E = \mathbf{E}(\Theta)$  such that first,  $E \subset \mathcal{A}$ , and second,

$$\sup_{\theta \in e} \bar{f}(\theta) - \inf_{\theta \in e} \bar{f}(\theta) < \varepsilon \quad \text{for all } e \in E.$$

Define

$$\bar{f}: \Theta \rightarrow \mathbb{R}, \quad \bar{f}(\theta) = \sup_{\theta \in e} \bar{f}(\theta) \quad \text{for all } \theta \in e \in E.$$

Clearly,

$$\bar{f} = \sum_{e \in E} \bar{f}(e) \cdot \mathbf{1}_e,$$

where

$$\bar{f}(e) = \{\bar{f}(\theta), \theta \in e\}$$

is a one-point set.

Therefore

$$\int f(\theta) \mu(d\theta) = \sum_{e \in E} \bar{f}(e) \mu(e) = \sum_{e \in E} \bar{f}(e) p(e) = \int \bar{f}(\theta) p(d\theta),$$

and at the same time,

$$\int \bar{f}(\theta) \mu(d\theta) \leq \int \bar{f}(\theta) \mu(d\theta) + \varepsilon \leq \int f(\theta) \mu(d\theta) + 2\varepsilon.$$

It remains to note that

$$\int f(\theta) p(d\theta) \leq \int \bar{f}(\theta) p(d\theta) = \int \bar{f}(\theta) \mu(d\theta)$$

and to let  $\varepsilon \rightarrow 0$ . We have thus proved Relation 1.

To prove Relation 2 note that if  $p \in PF(\Theta) \setminus P_\mu$ , then there exists  $A \in \mathcal{A}$  such that  $p(A) \neq \mu(A)$ . In this case, choosing

$$\varepsilon = \frac{|p(A) - \mu(A)|}{2}, \quad f = \mathbf{1}_A,$$

we obtain

$$U_{\varepsilon, f}(p) \cap P_\mu = \emptyset,$$

because

$$P'f = \mu(A) \quad \text{for all } p' \in P_\mu.$$

This proves Relation 2.

Relation 3 is proved in the following way. If the function  $f$  is measurable, then for every  $\varepsilon > 0$  we can choose a finite partition  $E \in \mathbf{E}(\Theta)$  of the set  $\Theta$  such that all  $e \in E$  are measurable, and

$$\sup_{\theta \in e} f(\theta) - \inf_{\theta \in e} f(\theta) < \varepsilon \quad \text{for all } e \in E.$$

We obtain

$$\left| \int f(\theta) \mu(d\theta) - \sum_{e \in E} \sup_{\theta \in e} f(\theta) \mu(e) \right| < \varepsilon,$$

$$\left| \int f(\theta) p(d\theta) - \sum_{e \in E} \sup_{\theta \in e} f(\theta) p(e) \right| < \varepsilon,$$

$p(e) = \mu(e)$ , for all  $e \in E$ , which immediately proves Relation 3.

It remains to note that given Relation 2 and the way of construction of the set  $P$  in the proof of the Relation 1, Relation 4 follows immediately from Lemma 5.2. So all the required relationships have been proved, and we easily obtain equalities (6.9)–(6.11), completing the proof of the theorem.

### 6.3 Optimality of Experiments

Note that informativity is not the only feature of observation that is relevant in practice. If it were so, we would always try to choose the observations only from the class  $H^*$ . In fact, the “complexity” of observation, i.e., the capacity of the set of observation values  $h(\Theta)$ , is important as well. Therefore, we introduce the following definition of an optimal observation:

**Definition 6.3.** An observation  $h \in H(\Theta)$  is called *optimal for the decision problem*  $S \in \mathbf{S}(\Theta)$  if

$$\text{INF}(h/S) \geq \text{INF}(h_1/S)$$

for any other observation  $h_1 \in H(\Theta)$ , and at the same time, if

$$\text{INF}(h_1/S) = \text{INF}(h/S),$$

then

$$\text{Card}(h(\Theta)) \leq \text{Card}(h_1(\Theta)),$$

where  $\text{Card}(A)$  is the capacity of the set  $A$ .

**Theorem 6.5.** *Every decision problem has an optimal observation, and every observation is optimal for some decision problem.*

*Proof.* Denote by  $H(S)$  the class of observations that have the highest informativity with respect to the decision problem  $S \in \mathbf{S}(\Theta)$ :

$$H(S) = \{h \in H(\Theta) : \text{INF}(h/S) = v(S)\}.$$

A certain capacity of the set  $h(\Theta)$  corresponds to every experiment  $h \in H(\Theta)$ , and we can consider the set of all these capacities

$$\{\text{Card}(h(\Theta)); h \in H(S)\}.$$

We know [2, Chapter 3, Theorem 2] that every set of cardinalities is completely ordered by value, and therefore there exists an observation  $h \in H(S)$  for which the capacity of the set  $h(\Theta)$  is minimal. This observation is obviously the optimal observation for the decision problem  $S$ .

Conversely, let  $h \in H(\Theta)$ . Assume that

$$U = h(\Theta) \quad L(\theta, u) = \begin{cases} 0, & \text{if } h(\theta) = u, \\ 1, & \text{if } h(\theta) \neq u, \end{cases}$$

$$P = PF(\Theta), \quad S = (\Theta, U, L, P).$$

First consider the case  $h \in H_*$ . Then the set  $U$  consists of a single element, and  $L(\theta, u) = 0$ ,  $\rho(S) = 0$ ,  $v(S) = 0$ . Thus the observation is optimal by definition. Now let  $h(\Theta)$  have more than one element. Then it is easy to show that

$$\rho(S) = \inf_{u \in U} \sup_{\theta \in \Theta} L(\theta, u) = 1,$$

$$\rho(S^h) = \inf_{d \in D} \sup_{\theta \in \Theta} L(\theta, d(h(\theta))) \leq \sup_{\theta \in \Theta} L(\theta, d_0(h(\theta))) = 0,$$

where

$$D = \{(h(\Theta) \rightarrow U) = \{U \rightarrow U\},$$

and  $d_0$  is the identity transformation of the set  $U$ . Since

$$\rho(S^h) \geq \inf_{\substack{\theta \in \Theta \\ u \in U}} L(\theta, u) = 0,$$

we obtain

$$\rho(S^h) = 0, \quad \text{INF}(h/S) = 1.$$

It is easy to verify that  $v(S) = 1$  as well, so that here it remains only to prove that if

$$h_1 \in H(S),$$

then

$$\text{Card}(h_1(\Theta)) \geq \text{Card}(h(\Theta)).$$

Let  $h_1 \in H(S)$ . Then  $\rho(S^{h_1}) = 0$ , and thus for every  $\varepsilon > 0$  there exists  $d_1 \in (h_1(\Theta) \rightarrow U)$  such that

$$\sup_{\theta \in \Theta} L(\theta, d_1(h_1(\theta))) < \varepsilon.$$

But the function  $L$  takes only the values 0 and 1, and thus

$$L(\theta, d_1(h_1(\theta))) = 0,$$

i.e.,

$$d_1(h_1(\theta)) = h(\theta) \quad \text{for all } \theta \in \Theta.$$

In other words, there exists a mapping  $d_1$  of the set  $h_1(\Theta)$  onto the set  $h(\Theta)$ . But this is possible only if the capacity of the set  $h(\Theta)$  does not exceed the capacity of the set  $h_1(\Theta)$ . The theorem is proved.

**Theorem 6.6.** *Let*

$$S = (\Theta, U, L, P) \in \mathbf{S}(\Theta), \quad h \in H(\Theta),$$

*be the optimal observation for  $S$ , and suppose the set  $U$  is finite. Then*

$$\text{Card}(h(\Theta)) \leq \text{Card}(U).$$

*Proof.* Since the set  $U$  is finite, we can associate to every  $\theta \in \Theta$  a nonempty subset of the set  $U$  on which the function  $L(\theta, \cdot)$  takes its minimal value:

$$U_\theta = \arg \min_{u \in U} L(\theta, u).$$

Let

$$\bigcup_{\theta \in \Theta} U_\theta = U^\theta = \{u_1, u_2, \dots, u_n\}.$$

Set

$$\Theta_1 = \{\theta \in \Theta : U_\theta \ni u_1\},$$

$$\Theta_i = \left\{ \theta \in \Theta : U_\theta \ni u_i, \theta \in \bigcup_{j=1}^{i-1} \Theta_j \right\} \quad (i = 2, 3, \dots, n).$$

Clearly,

$$\bigcup_{i=1}^n \Theta_i = \Theta \quad \text{and} \quad \Theta_i \cap \Theta_j = \emptyset \quad \text{for } i \neq j.$$

Let

$$h_1 : \Theta \rightarrow \{1, 2, \dots, n\}, \quad h_1(\theta) = i \quad \text{for } \theta \in \Theta_i.$$

We have

$$\rho(S^{h_1}) = \inf_{d \in D} \max_{p \in P} \int L(\theta, d(h_1(\theta))) p(d\theta) = \max_{p \in P} \int \left[ \min_{u \in U} L(\theta, u) \right] p(d\theta).$$

i.e.,

$$\text{INF}(h_1/S) = v(S).$$

If some experiment  $h$  is optimal for  $S$ , then

$$\text{Card}(h(\Theta)) \leq \text{Card}(h_1(\Theta)) \leq n \leq \text{Card}(U).$$

The theorem is proved.

# Chapter 7

## Informativity of Experiment in Bayesian Decision Problems

While we feel that information theory is indeed a valuable tool in providing fundamental insights into the nature of communication problems and will continue to grow in importance, it is certainly no panacea for the communication engineer or, a fortiori, for anyone else.

---

Claude Shannon

### 7.1 Preliminaries

In this chapter we shall study the question of interconnection between the concept of informativity of experiment and Shannon's concept "quantity of information," contained in this experiment. Here the class of decision problems and experiments is significantly narrowed with respect to previous chapters. Therefore it is convenient to change slightly the notation and terminology.

Let  $\Theta$  and  $W$  be arbitrary finite sets,  $\mathcal{P}(\Theta)$  and  $\mathcal{P}(W)$  families of all probability distributions on  $\Theta$  and  $W$  respectively.

Let  $\eta = (\eta_\theta)_{\theta \in \Theta}$  be a random function of  $\theta$  taking values in  $W$  (stochastic experiment on  $\Theta$ ). By observing these values we obtain some information about  $\theta$ . How can we estimate the quantity of this information?

In the case in which the parameter  $\theta$  itself is random with a given distribution  $P \in \mathcal{P}(\Theta)$ , classical information theory [75] proposes to measure the quantity of information about  $\theta$ , contained in  $\eta$ , by the difference

$$I(P, \eta) = H(P) - H_\eta(P),$$

where

$$H(P) = - \sum_{\theta \in \Theta} P(\theta) \log_2 P(\theta) \quad (7.1)$$

is the entropy of the distribution  $P$ , and

$$H_\eta(P) = \sum_{w \in W} H(P(\cdot/\eta = w)) \sum_{\theta \in \Theta} P(\theta) \mu(w/\theta)$$

is the average entropy of the conditional (on  $\eta$ ) distribution of  $\theta$  ( $\mu(\cdot/\theta)$  is a distribution of the random element  $\eta_\theta$ ).

Suppose now that a stochastic experiment is conducted with the aim to improve decisions in the decision problem

$$S = (\Theta, U, L, P),$$

where  $U$  is a nonempty set of possible decisions, one of which  $u \in U$  we have to choose on the basis of the experiment;  $L$  is a real loss function

$$L : \Theta \times U \rightarrow \mathbb{R}$$

bounded from below. In this case, due to the experiment  $\eta$ , we manage to decrease the minimal average losses from the value

$$r(P) = \inf_{u \in U} \sum_{\theta \in \Theta} L(\theta, u) P(\theta)$$

to the value

$$r_\eta(P) = \sum_{\substack{\theta \in \Theta \\ w \in W}} P(\theta) \mu(w/\theta) \left[ \inf_{u \in U} \sum_{\theta' \in \Theta} L(\theta', u) P(\theta'/\eta = w) \right].$$

The inequality

$$r(P) \geq r_\eta(P)$$

follows formally from the concavity of the function  $r(\cdot)$ , and follows intuitively from the fact that the experiment supplies some information about  $\theta$ . Therefore the difference  $r(P) - r_\eta(P)$  is usually called the “value” of information, or the value of “inquiry” [62], contained in  $\eta$ . We shall denote it by  $\Delta(P, \eta, U, L)$ .

Thus if an a priori distribution  $P$  of the parameter  $\theta$  is fixed, any stochastic experiment  $\eta$  contains the quantity of information about  $\theta$  equal to  $I(P, \eta)$ , and if, in addition, the aim of the experiment is fixed in the form of  $U$  and  $L$ , then any stochastic experiment  $\eta$  has the value  $\Delta(P, \eta, U, L)$ . In this connection, as will be seen below, one can, generally speaking, find a pair of stochastic experiments  $\eta_1$  and  $\eta_2$  such that the value of the first will be larger, but the second will be more informative [32].

The question arises as to what the value of the experiment consists in if not in the fact that it contains some information about the unknown parameter. And if

that is so, how can an experiment containing less information be more valuable for us?

One might consider this as simply a terminological inaccuracy, but unfortunately, to correct it is not easy. The fact is that on the one hand, Shannon's "information quantity" plays such an important role in modern science that it is impossible to ignore. On the other hand, when the choice of the optimal experiment is made with the aim to reduce average losses, to subordinate this choice to any external factor would be a deliberate absurdity. Therefore, the only way in which one can overcome this difficulty is to assume that in different situations information has to be measured in different ways [76]. But what does the word "situation" mean here, and what is the arbitrariness in the choice of the method of measurement in every "situation"?

Before answering this question let us try to understand how this problem is connected to what was said above in Chapters 5 and 6. If we set

$$\begin{aligned}\tilde{\Theta} &= \Theta \times W, \\ p(A) &= \sum_{(\theta, w) \in A} P(\theta) \mu(w/\theta) \quad \text{for all } A \subseteq \tilde{\Theta}, \\ h: \tilde{\Theta} &\rightarrow W, \quad h(\theta, w) = w, \\ \tilde{L}: \tilde{\Theta} \times U &\rightarrow \mathbb{R}, \\ \tilde{L}((\theta, w), u) &= L(\tilde{\theta}, u),\end{aligned}$$

then we see that

$$S = (\tilde{\Theta}, U, \tilde{L}, \{p\})$$

is a Bayesian decision problem in our usual sense of the term,  $h$  is the observation on  $\tilde{\Theta}$ ,  $r(P) = \rho(S)$  is the risk of this decision problem, while the risk after observation is

$$r_{\eta}(P) = \rho(S^h), \quad (7.2)$$

the value of information is

$$\Delta(P, \eta, U, L) = \text{INF}(h/S), \quad (7.3)$$

and the uncertainty of the decision problem (see Section 6.1) is

$$v(S) = \inf_{u \in U} \sum_{\tilde{\theta} \in \tilde{\Theta}} L(\tilde{\theta}, u) P(\tilde{\theta}) - \sum_{\tilde{\theta} \in \tilde{\Theta}} [\inf_{u \in U} L(\tilde{\theta}, u)] P(\tilde{\theta}). \quad (7.4)$$

Now, returning to the question about the possible ways of information measurement, we introduce some auxiliary notation. Let

$$\gamma: \mathcal{P}(\Theta) \rightarrow \mathbb{R}$$

be an arbitrary real function, suppose  $P \in \mathcal{P}(\Theta)$ , and let  $\eta$  be a stochastic experiment on  $\Theta$  with finite set of values  $W$ . Denote by  $I(P, \eta, \gamma)$  the average decrease of the function  $\gamma$  as a result of the experiment  $\eta$ , i.e.,

$$I(P, \eta, \gamma) = \gamma(P) - \sum_{w \in W} \gamma(P(\cdot/\eta = w)) \sum_{\theta \in \Theta} P(\theta) \mu(w/\theta). \quad (7.5)$$

In particular, one can now write Shannon's quantity of information  $I(P, \eta)$  in the form  $I(P, \eta, H)$ .

Our first goal is to understand what properties the function  $\gamma$  should possess in order that one could characterize the information quantity contained in the experiment  $\eta$  by its average decrease  $I(P, \eta, \gamma)$ .

**Definition 7.1.** Let  $Z = (\Theta, U, L)$  be an arbitrary decision scheme with finite set  $\Theta$ . Set  $n: P(\Theta) \rightarrow \mathbb{R}$ . We say that the function  $n$  is the *natural uncertainty function* of the decision scheme  $Z$  if it satisfies the following two conditions:

(1) For any two statistical experiments  $\eta_1$  and  $\eta_2$  on  $\Theta$  we have the relation

$$\text{sgn}[I(P, \eta_1, n) - I(P, \eta_2, n)] = \text{sgn}[r_{\eta_2}(P) - r_{\eta_1}(P)], \forall P \in \mathcal{P}(\Theta).$$

(2) If the distribution  $P$  is concentrated at one point, then  $n(P) = 0$ .

It is easy to verify that the uncertainty  $v(S)$ , considered as a function of  $P$  (see (7.4)), is one of the natural uncertainty functions. Therefore condition (2) means that in the Bayesian case, we want to keep the property of the uncertainty equal to zero when  $\theta$  is known exactly. Condition (1) means that the paradoxical situation in which it would be more advantageous to produce an experiment containing less information cannot arise. It turns out that the definition just formulated assigns the natural uncertainty function univalently to within the scale. Namely (Theorem 7.1), those and only those functions that are representable in the form  $n(P) = \lambda n_Z(P)$ , where  $\lambda > 0$  and

$$n_Z(P) = \inf_{u \in U} \sum_{\theta \in \Theta} L(\theta, u) P(\theta) - \sum_{\theta \in \Theta} [\inf_{u \in U} L(\theta, u)] P(\theta),$$

can serve as a natural uncertainty function for the decision scheme  $Z = (\Theta, U, L)$ . In other words, if the decision problem  $(\Theta, U, L, P)$  is denoted by  $S$ , we obtain

$$n_Z(P) = v(S), \quad n(P) = \lambda v(S).$$

But since the decrease of the uncertainty, as was already mentioned, is the "usefulness" or the "value" of information, Theorem 7.1 can be interpreted in a way that once the aim of the experiment (decision scheme  $Z$ ) is fixed, there

is no other method of measuring the quantity of information but to measure its usefulness (i.e., informativity).

This immediately implies the following question: is it possible to find “the aim of the experiment” leading to the method of information measurement introduced by Shannon?

To answer this question suppose first that the aim is not fixed and call any function  $n : \mathcal{P}(\Theta) \rightarrow \mathbb{R}$  satisfying the following conditions an *uncertainty function on  $\mathcal{P}(\Theta)$* :

- (1)  $I(P, \eta, n) \geq 0$  for any stochastic experiment (see (7.5));
- (2)  $n(P) = 0$  if  $P$  is concentrated at one point;
- (3) the function  $n$  is continuous in the natural topology of the set  $\mathcal{P}(\Theta)$ .

It turns out that for any uncertainty function  $n$  on  $\mathcal{P}(\Theta)$  one can find a decision scheme  $Z = (\Theta, U, L)$  for which this function will be the natural uncertainty function (Theorem 7.2).

In other words, even if the aim is not indicated explicitly, any “reasonable” method of measuring information quantity explicitly presupposes such an aim [35].

In particular, Shannon’s entropy (7.1) is the natural uncertainty function of the decision scheme  $Z = (\Theta, U, L)$ , where  $U = \mathcal{P}(\Theta)$  and  $L(\Theta, u) = -\log u(\theta)$ . One can interpret this decision scheme as a problem of pattern recognition, described, for instance, in [6]. We rewrite it here in a form that is convenient for us.

Let there be given a finite set  $M$  of objects, divided into equivalence classes. The set of equivalence classes is denoted by  $\Theta$ . Further let the probability distribution  $p$ , inducing the distribution  $P$  on  $\Theta$ , be given on  $M$ :

$$P(\theta) = \sum_{m \in \theta} p(m) \quad (\theta \in \Theta).$$

The objects  $m \in M$  are presented to us independently one by one with probability  $p(m)$ , and we must associate to every object the equivalence class  $\theta_m$  containing this object. We do not know beforehand which class the presented object belongs to. We are allowed to use only one method of recognition: having chosen some probability distribution  $u$  on  $\theta$ , we randomly sample this distribution, and then, by means of the special algorithm, we check whether the presented object belongs to the class that resulted from the random sample. In case of failure, we sample again, while in case of success, the next object is shown. The distribution  $u \in \mathcal{P}(\Theta)$  does not change during the whole process of recognition.

Denote by  $l(m, u)$  the average number of failures (the number of tests before the first successful test) for a given object  $m$  and for a fixed probability distribution  $u \in \mathcal{P}(\Theta) = U$  of the random sample. Obviously,

$$l(m, u) = 1u(\theta_m) + 2(1 - u(\theta_m))u(\theta_m) + \dots = \frac{1}{u(\theta_m)},$$

and if we describe the quality of recognition by mathematical expectation of the value  $\log l(m, n)$ , i.e., by the number

$$\sum_{m \in M} \log \frac{1}{u(\theta_m)} p(m) = \sum_{\theta \in \Theta} \sum_{m \in M} \log \frac{1}{u(\theta)} p(m) = - \sum_{\theta \in \Theta} \log u(\theta) p(\theta)$$

(serious reasons for this are presented in [6]), we arrive at the decision problem described above.

Perhaps another decision problem is more interesting. Let  $\Theta$  be the set of symbols to be encoded,  $U$  the set of all possible binary codes, and  $L(\theta, u)$  the length of the “word” corresponding to the symbol  $\theta \in \Theta$  when the code is  $u \in U$ . However, it may be more appealing in case of the encoding decision problem to consider as loss function the excess length defined as in [14]:

$$\hat{L}(\theta, u) = L(\theta, u) - \min_{u \in U} L(\theta, u).$$

Since

$$\min_u \hat{L}(\theta, u) = \min_u (L(\theta, u) - \min_u L(\theta, u)) = 0,$$

then

$$\begin{aligned} v_{\hat{L}}(P) &= \min_u \sum_{\theta} \hat{L}(\theta, u) p(\theta) - \sum_{\theta} \min_u \hat{L}(\theta, u) p(\theta) \\ &= \min_u \sum_{\theta} L(\theta, u) p(\theta) - \sum_{\theta} \min_u L(\theta, u) p(\theta) = v_L(P), \end{aligned}$$

with  $v_L(P)$  being the uncertainty function of the decision system with the loss function  $L(\theta, u)$ . That is, the uncertainty functions of these two decision systems are the same:

$$v_{\hat{L}}(P) = v_L(P).$$

On the other hand,

$$v_{\hat{L}}(P) = \min_u \sum_{\theta} \hat{L}(\theta, u) p(\theta).$$

We now prove that

$$\min_u \sum_{\theta} \hat{L}(\theta, u) p(\theta) = H(P),$$

where  $H(P)$  is the entropy of the distribution  $P$ .

Let  $\theta_i \in \Theta$ ,  $i = 1, \dots, N$ . We set  $\hat{L}(\theta_i, u) = d_u(\theta_i) = d_i^u$ , the excess length of the symbol  $\theta_i \in \Theta$  in the code  $u \in U$ . Let  $\{d_i^u\}, i = 1, \dots, N$  be the collection

of excess lengths of the symbols in the code  $u$ . Let  $D$  be the set of all such collections. Obviously, the set  $U$  is isomorphic to the set  $D$ . Therefore,

$$\min_{u \in U} \sum_{\theta} \hat{L}(\theta, u) p(\theta) = \min_{u \in U} \sum_{i=1}^N d_i^u p_i = \min_{d \in D} \sum_{i=1}^N d_i p_i.$$

Now consider the following minimization problem (see [85] for details):

$$\sum_{i=1}^N d_i p_i \rightarrow \min, \quad \sum_{i=1}^N p_i = 1, \quad \sum_{i=1}^N B^{-d_i} \leq 1, \quad d_i > 0, \quad \forall i = 1, 2, \dots, n,$$

where  $B$  is a positive integer, interpreted in this case as the base of the encoding system (if  $B = 2$  we have the binary system).

Using the technique of Lagrange multipliers we have

$$\begin{aligned} \mathcal{L}(d_1, \dots, d_N, \lambda) &= \sum_{i=1}^N d_i p_i + \lambda \left( 1 - \sum_{i=1}^N B^{-d_i} \right), \\ \frac{\partial \mathcal{L}}{\partial d_i} &= p_i + \lambda B^{-d_i} \ln B = 0, \end{aligned}$$

whence taking the sum over  $i = 1, \dots, N$ , we obtain

$$1 + \lambda \ln B = 0, \quad \lambda = -\frac{1}{\ln B},$$

or

$$p_i - B^{-d_i} = 0, \quad d_i = -\log_B p_i,$$

which means that

$$\min_{d \in D} \sum_{i=1}^N d_i p_i = -\sum_{i=1}^N p_i \log_B p_i = H(P).$$

Thus

$$v_{\hat{L}}(P) = v_L(P) = H(P),$$

where  $H(P)$  is, on the one hand, the entropy of the distribution  $P$ , while on the other, it is a quantitative characteristic of the informativity of the experiment in the decision problem about the optimal code, with the length (or excess length) of the code as loss function: another loss function, as was stated by Shannon in [76], would not necessarily result in the entropy as uncertainty measure. This can be a justification of the fact that entropy as uncertainty measure appeared for the first time in communication problems.

Summarizing the obtained results, one can say that a decision scheme  $Z = (\Theta, U, L)$  predetermines the information measurement method, and together with the “a priori information on the behavior of  $\theta$ ,” the distribution  $P$ , constitutes the complete mathematical description of the situation in which the decision is to be made. Thus relative to the decision problem  $S = (\Theta, U, L, P)$ , the quantity of information contained in any stochastic experiment (with finite set of values) can be found uniquely to within the units of measurement.

## 7.2 Theorem of Existence and Uniqueness of Uncertainty Functions

In what follows we suppose that the set  $\Theta$  is finite and for any set  $X$  denote by  $\mathcal{P}(X)$  the family of all probability distributions on it, and by  $\mathcal{P}_X$  the family of all such  $P \in \mathcal{P}(X)$  that are concentrated in one point. By a *stochastic experiment* on  $\Theta$  we mean any regular pair  $\eta = (W, \mu)$ , where  $W$  is some finite set and  $\mu$  the transition probability from  $\Theta$  in  $W$ , i.e.,

$$\mu : \Theta \times W \rightarrow [0, 1], \quad (\theta, w) \rightarrow \mu(w/\theta), \quad \sum_{w \in W} \mu(w/\theta) = 1.$$

A stochastic experiment on  $\Theta$  will be denoted further by  $\eta$  with or without indices.

Now associate to each

$$Z = (\Theta, U, L) \in \mathbb{Z}(\Theta)$$

the function

$$\gamma_Z : \mathcal{P}(\Theta) \rightarrow \mathbb{R}, \quad \gamma_Z(P) = \inf_{\mu \in U} \sum_{\theta \in \Theta} L(\theta, u) P(\theta),$$

and then to each triple of the form  $(\eta, f, P)$ , where  $\eta$  is a stochastic experiment on  $\Theta$ , and  $f : \mathcal{P}(\Theta) \rightarrow \mathbb{R}$ ,  $P \in \mathcal{P}(\Theta)$ , we associate the real number

$$I(\eta, f, P) = f(P) - \sum_{w \in W} p(w; P, \eta) f(P_\eta(\cdot/w)),$$

where

$$p(w; P, \eta) = \sum_{\theta \in \Theta} P(\theta) \mu(w/\theta),$$

$$P_\eta(\theta/w) = \begin{cases} \frac{P(\theta) \mu(w/\theta)}{p(w; P, \eta)}, & \text{if } p(w; P, \eta) \neq 0, \\ \frac{1}{\text{Card}(\Theta)}, & \text{if } p(w; P, \eta) = 0. \end{cases}$$

**Definition 7.2.** Let  $Z = (\Theta, U, L)$  be an arbitrary decision scheme with finite set  $\Theta$ ,  $n : \mathcal{P}(\Theta) \rightarrow \mathbb{R}$ . We say that the function  $n$  is the *natural uncertainty function of the decision scheme  $Z$*  if it satisfies the following two conditions:

**Condition 1** For any two stochastic experiments  $\eta_1$  and  $\eta_2$  on  $\Theta$ , we have

$$\text{sgn}[I(\eta_1, n, P) - I(\eta_2, n, P)] = \text{sgn}[I(\eta_2, \gamma_Z, P) - I(\eta_1, \gamma_Z, P)]$$

for all  $P \in \mathcal{P}(\Theta)$ .

**Condition 2** If the distribution  $P$  is concentrated at one point, i.e.,  $P \in \mathcal{P}_\Theta$ , then  $n(P) = 0$ .

The set of all natural uncertainty functions of the decision scheme  $Z$  will be denoted by  $\mathfrak{N}(Z)$ .

**Theorem 7.1.** For the decision scheme  $Z = (\Theta, U, L)$ , the functions that can serve as natural uncertainty functions are those that can be represented in the form

$$\begin{aligned} n(P) &= \lambda n_Z(P), \quad \text{where } \lambda > 0, \\ n_Z(P) &= \inf_{u \in U} \sum_{\theta \in \Theta} L(\theta, u) P(\theta) - \sum_{\theta \in \Theta} [\inf_{u \in U} L(\theta, u)] P(\theta). \end{aligned}$$

In other words, if we denote the decision problem  $(\Theta, U, L, P)$  by  $S$ , we obtain

$$n_Z(P) = v(S), \quad n(P) = \lambda v(S).$$

*Proof.* It is convenient to divide the proof of this theorem into several stages. Let  $E$  be some linear space,  $A \subseteq E$  a convex set,  $A \subseteq B \subseteq E$ , and  $f : B \rightarrow \mathbb{R}$ . We say that the function  $f$  is affine (respectively concave) on  $A$  if for any  $x_1, x_2 \in A$  we have

$$f((1 - \alpha)x_1 + \alpha x_2) = (1 - \alpha)f(x_1) + \alpha f(x_2) \quad \forall \alpha \in [0, 1]$$

(respectively  $f((1 - \alpha)x_1 + \alpha x_2) \geq (1 - \alpha)f(x_1) + \alpha f(x_2) \quad \forall \alpha \in [0, 1]$ ).

(1) We first prove that if

$$\begin{aligned} a, b, c, d, c_n, d_n &\in \mathbb{R} \quad \forall n \in \mathbb{N}, \\ a < c < d < b, \quad c_n &\rightarrow c, \quad d_n \rightarrow d \quad (n \rightarrow \infty) \end{aligned}$$

and the function  $f$  is concave on  $[a, b]$ , then from the affinity of  $f$  on all segments  $[c_n, d_n]$  beginning from some  $n = n_0$  follows the affinity of  $f$  on  $[c, d]$ .

According to [5], a bounded concave<sup>1</sup> function  $f$  is continuous on  $]a, b[$ . Therefore, beginning from some  $n \geq n_0$ , it is also continuous on  $[c_n, d_n]$ , whence

<sup>1</sup> Not concave but convex functions are considered in [5], as is usual in the literature. Therefore, let us agree here that when we give references, we shall not stipulate every time the obvious changes that have to be made in the presentation.

for every  $x \in ]c_n, d_n[$ , it follows that

$$f(x) = \frac{d_n - x}{d_n - c_n} f(c_n) + \frac{x - c_n}{d_n - c_n} f(d_n).$$

Passage to the limit  $n \rightarrow \infty$  completes the proof.

(2) Now let

$$x_1, x_2, x_3, x_4 \in \mathbb{R}, \quad x_1 x_4 \neq x_2 x_3.$$

We shall show that there exists  $r \neq 0$  such that

$$(x_1 + r x_2)(x_3 + r x_4) < 0.$$

Rewrite this inequality in the form

$$r^2 x_2 x_4 + r(x_1 x_4 + x_2 x_3) + x_1 x_3 < 0.$$

The discriminant of the left side (as a quadratic trinomial in  $r$ ) is equal to

$$(x_1 x_4 - x_2 x_3)^2 > 0;$$

hence the desired values of  $r$  fill the entire interval between the roots of this trinomial.

(3) For an arbitrary convex set  $A \subset E$  let us write in the form  $(f_1, f_2) \in \mathfrak{L}(A)$  the following statement:  $f_1$  and  $f_2$  are two concave functions on the set  $A$ , and for any

$$\begin{aligned} x_1, x_2, \dots, x_m, & \quad x'_1, x'_2, \dots, x'_n \in A; \\ \alpha_1, \alpha_2, \dots, \alpha_m, & \quad \alpha'_1, \alpha'_2, \dots, \alpha'_n \in [0, 1], \end{aligned}$$

from

$$\sum_{i=1}^m \alpha_i = \sum_{j=1}^n \alpha'_j = 1, \quad \sum_{i=1}^m \alpha_i x_i = \sum_{j=1}^n \alpha'_j x'_j,$$

it follows that

$$\operatorname{sgn} \left[ \sum_{i=1}^m \alpha_i f_1(x_i) - \sum_{j=1}^n \alpha'_j f_1(x'_j) \right] = \operatorname{sgn} \left[ \sum_{i=1}^m \alpha_i f_2(x_i) - \sum_{j=1}^n \alpha'_j f_2(x'_j) \right].$$

We shall show that if  $(f_1, f_2) \in \mathfrak{L}(A)$  and the function  $f_1$  is affine on  $A$ , then the function  $f_2$  is affine on  $A$  as well. Indeed, let

$$x_1, x_2 \in A, \quad \alpha \in ]0, 1[, \quad x_3 = (1 - \alpha)x_1 + \alpha x_2.$$

According to our condition,

$$f_1(x_3) = (1 - \alpha)f_1(x_1) + \alpha f_1(x_2).$$

Define

$$\alpha_1 = \alpha'_1 = 1 - \alpha, \quad \alpha_2 = \alpha'_2 = \alpha, \quad x'_1 = x'_2 = x_3.$$

Then

$$\operatorname{sgn} \left[ \sum_{i=1}^2 \alpha_i f_1(x_i) - \sum_{j=1}^2 \alpha'_j f_1(x'_j) \right] = 0,$$

so the same equality will be true if  $f_1$  is replaced by  $f_2$ . The arbitrariness of  $x_1$ ,  $x_2$ , and  $\alpha$  implies the affineness of  $f$ .

(4) The following fact will be necessary in what follows. If the function  $f$  is concave and not affine on the segment  $[a, b] \subset \mathbb{R}$ , then

$$f(c) > \frac{b-c}{b-a}f(a) + \frac{c-a}{b-a}f(b) \quad \text{for all } c \in ]a, b[.$$

To prove this, note that from the condition on  $f$  it follows immediately that there exists  $x \in ]a, b[$  such that

$$f(x) > \frac{b-x}{b-a}f(a) + \frac{x-a}{b-a}f(b).$$

If  $x = c$ , all is proved, and the cases  $x < c$  and  $x > c$  are considered analogously. Therefore we prove only the first of them. From the concavity of  $f$  we have

$$f(c) \geq f(x) \frac{b-c}{b-x} + f(b) \frac{c-x}{b-x},$$

whence, with regard to the previous inequality,

$$\begin{aligned} f(c) &> \left[ \frac{b-x}{b-a}f(a) + \frac{x-a}{b-a}f(b) \right] \frac{b-c}{b-x} + f(b) \frac{c-x}{b-x} \\ &= f(a) \frac{b-c}{b-a} + f(b) \left[ \frac{(x-a)(b-c)}{(b-a)(b-x)} + \frac{(c-x)}{(b-x)} \right] \\ &= f(a) \frac{b-c}{b-a} + f(b) \frac{c-a}{b-a}. \end{aligned}$$

From here it follows that under the same conditions, the inequalities

$$\frac{f(c) - f(a)}{c - a} > \frac{f(b) - f(a)}{b - a} > \frac{f(b) - f(c)}{b - c}$$

are satisfied. Indeed,

$$\frac{f(c) - f(a)}{c - a} > \frac{\frac{b-c}{b-a}f(a) + \frac{c-a}{b-a}f(b) - f(a)}{c - a} = \frac{f(b) - f(a)}{b - a}.$$

The second inequality is proved analogously. One more obvious consequence of the concavity and nonaffineness of the function  $f$  on the segment  $[a, b]$  is that

$$f^{(r)}(a) > \frac{f(b) - f(a)}{b - a} > f^{(l)}(b),$$

where  $f^{(r)}(\cdot)$  and  $f^{(l)}(\cdot)$  are the right and left derivatives of the function  $f$ .

(5) Now we are ready to prove the following important lemma.

**Lemma 7.1.** *Let  $(f_1, f_2) \in \mathfrak{L}([a, b])$ ,  $c_i$  the set of points from the interval  $]a, b[$  in which the function  $f_i$  ( $i = 1, 2$ ) is differentiable. Let further  $x_1, x_2 \in c_1 \cap c_2$ ,  $x_1 < x_2$ , and suppose that the function  $f_1$  (and hence  $f_2$ ) is not affine on  $[x_1, x_2]$ . Set*

$$G_i(x_1, x_2) = \frac{f'_i(x_1) - \frac{f_i(x_2) - f_i(x_1)}{x_2 - x_1}}{f'_i(x_2) - \frac{f_i(x_2) - f_i(x_1)}{x_2 - x_1}} \quad (i = 1, 2).$$

Then

$$G_1(x_1, x_2) = G_2(x_1, x_2) \neq \pm\infty.$$

*Proof.* Take an arbitrary  $]x_1, x_2[$  and choose the functions  $F_1$  and  $F_2$  on the direct product of intervals  $]a, x_0[$  and  $]x_0, b[$  defined by the following relationship:

$$F_i(x, y) = \frac{y - x_0}{y - x} f_i(x) + \frac{x_0 - x}{y - x} f_i(y) \quad (i = 1, 2).$$

It is easy to show that this relationship can be described in either of the two following forms:

$$F_i(x, y) = f_i(x_2) + \frac{x_2 - x_0}{x_2 - x_1} [f_i(x_1) - f_i(x_2)],$$

$$F_i(x, y) = f_i(x_1) + \frac{x_0 - x_1}{x_2 - x_1} [f_i(x_2) - f_i(x_1)].$$

Therefore, from the differentiability of the functions  $f_i$  at the points  $x_1$  and  $x_2$  follows the existence of the partial derivatives

$$\begin{aligned}\frac{\partial F_i(x_1, x_2)}{\partial x_2} &= \frac{x_2 - x_0}{x_2 - x_1} f'_i(x_1) + \frac{x_2 - x_0}{(x_2 - x_1)^2} [f_i(x_1) - f_i(x_2)] \\ &= \frac{x_2 - x_0}{x_2 - x_1} \left[ f'_i(x_1) - \frac{f_i(x_2) - f_i(x_1)}{x_2 - x_1} \right], \\ \frac{\partial F_i(x_1, x_2)}{\partial x_1} &= \frac{x_0 - x_1}{x_2 - x_1} f'_i(x_2) - \frac{x_0 - x_1}{(x_2 - x_1)^2} [f_i(x_2) - f_i(x_1)] \\ &= \frac{x_0 - x_1}{x_2 - x_1} \left[ f'_i(x_2) - \frac{f_i(x_2) - f_i(x_1)}{x_2 - x_1} \right],\end{aligned}$$

which, as we noted above, are not equal to zero.

Thus,

$$G_1(x_1, x_2) = \frac{x_0 - x_1}{x_2 - x_1} \frac{\frac{\partial F_i(x_1, x_2)}{\partial x_1}}{\frac{\partial F_i(x_1, x_2)}{\partial x_2}},$$

and if the statement of the lemma were not valid, we would have

$$\frac{\partial F_1(x_1, x_2)}{\partial x_1} \frac{\partial F_2(x_1, x_2)}{\partial x_2} \neq \frac{\partial F_1(x_1, x_2)}{\partial x_2} \frac{\partial F_2(x_1, x_2)}{\partial x_1}.$$

In this case, as we saw,  $r \neq 0$  could be found such that we would have the inequality

$$\left[ \frac{\partial F_1(x_1, x_2)}{\partial x_1} + r \frac{\partial F_1(x_1, x_2)}{\partial x_2} \right] \left[ \frac{\partial F_2(x_1, x_2)}{\partial x_1} + r \frac{\partial F_2(x_1, x_2)}{\partial x_2} \right] < 0. \quad (7.6)$$

Our aim is to show that this would result in  $(f_1, f_2) \notin \mathfrak{U}([a, b])$ , contrary to the statement of the lemma. To do this we shall try to choose  $t > 0$  such that the following four conditions are satisfied:

**Condition 1**  $x_1 + t \stackrel{\text{def}}{=} x'_1 \in C_1 \cap C_2 \cap ]a, x_0[;$

**Condition 2**  $x_2 + rt \stackrel{\text{def}}{=} x'_2 \in C_1 \cap C_2 \cap ]x_0, b[;$

**Condition 3** *the functions  $f_1$  and  $f_2$  are not affine on  $[x'_1, x'_2]$ ;*

**Condition 4**

$$\text{sgn} [F_1(x'_1, x'_2) - F_1(x_1, x_2)] \neq \text{sgn} [F_2(x'_1, x'_2) - F_2(x_1, x_2)].$$

From the concavity of the function  $f_i$  it follows that the sets  $[a, b] \setminus C_i$  are no more than countable [5], and so “nearly all”  $t$  such that  $x'_1 \in ]a, x_0[$ ,  $x'_2 \in ]x_0, b[$  satisfy Conditions 1 and 2 simultaneously. From this we conclude that simultaneously with Conditions 1 and 2 one can also satisfy Condition 3, since

otherwise, selecting the corresponding sequence  $t_n \searrow 0$ , we would arrive at the affineness of the functions  $f_1$  and  $f_2$  on  $[x_1, x_2]$ .

It remains to show that for sufficiently small  $t$ , Condition 4 follows from Conditions 1–3. Indeed, if Conditions 1–3 are satisfied, the numbers  $x'_1$  and  $x'_2$  in the statement of the lemma may play the role of the numbers  $x_1$  and  $x_2$ , and so, according to what we have already proved, the partial derivatives

$$\frac{\partial F_1(x'_1, x'_2)}{\partial x'_2} \quad \text{and} \quad \frac{\partial F_2(x'_1, x'_2)}{\partial x'_2}$$

exist. Moreover, by virtue of the continuity of the concave functions  $f_1$  and  $f_2$  on  $]a, b[$ ,

$$\lim_{\substack{x'_1 \rightarrow x \\ x'_1 \in C_i}} \frac{\partial F_i(x'_1, x'_2)}{\partial x_2} = \frac{\partial F_i(x_1, x_2)}{\partial x_2} \quad i = (1, 2). \quad (7.7)$$

Therefore, if we denote by  $T$  the set of all  $t > 0$  such that Conditions 1, 2, and 3 are satisfied simultaneously, we obtain that when  $t \in T$ ,  $t \rightarrow 0$ , then

$$\begin{aligned} & F_i(x'_1, x'_2) - F_i(x_1, x_2) \\ &= [F_i(x'_1, x'_2) - F_i(x'_1, x_2)] + [F_i(x'_1, x_2) - F_i(x_1, x_2)] \\ &= \frac{\partial F_i(x'_1, x_2)}{\partial x_2} (x'_2 - x_2) + \frac{\partial F_i(x_1, x_2)}{\partial x_1} (x'_1 - x_1) + 0(t) \\ &= \left[ \frac{\partial F_i(x_1, x_2)}{\partial x_1} + r \frac{\partial F_i(x'_1, x_2)}{\partial x_2} \right] t + 0(t) \\ &= \left[ \frac{\partial F_i(x_1, x_2)}{\partial x_1} + r \frac{\partial F_i(x'_1, x_2)}{\partial x_2} \right] t + 0(t) + rt \left[ \frac{\partial F_i(x'_1, x_2)}{\partial x_2} - \frac{\partial F_i(x_1, x_2)}{\partial x_2} \right]. \end{aligned}$$

Therefore by virtue of (7.6) and (7.7), if  $t \in T$  are sufficiently small, then

$$\begin{aligned} \text{sgn} [F_1(x'_1, x'_2) - F_1(x_1, x_2)] &= \text{sgn} \left[ \frac{\partial F_1(x_1, x_2)}{\partial x_1} + r \frac{\partial F_1(x_1, x_2)}{\partial x_2} \right] \\ &\neq \text{sgn} \left[ \frac{\partial F_2(x_1, x_2)}{\partial x_1} + r \frac{\partial F_2(x_1, x_2)}{\partial x_2} \right] \\ &= \text{sgn} [F_2(x'_1, x'_2) - F_1(x_1, x_2)]. \end{aligned}$$

In other words, we have proved the existence of the required numbers  $t$ . It remains only to note that when

$$\alpha_1 = \frac{x_2 - x_0}{x'_2 - x'_1}, \quad \alpha_2 = \frac{x_0 - x_1}{x_2 - x_1},$$

$$\alpha'_1 = \frac{x'_2 - x_0}{x'_2 - x'_1}, \quad \alpha'_2 = \frac{x_0 - x'_1}{x'_2 - x'_1},$$

the condition for  $(f_1, f_2) \in \mathfrak{U}([a, b])$  in particular requires the relationship

$$\operatorname{sgn} [F_1(x_1, x_2) - F_1(x'_1, x'_2)] = \operatorname{sgn} [F_2(x_1, x_2) - F_2(x'_1, x'_2)],$$

so we have obtained a contradiction, which proves the lemma.

(6) We prove the following lemma.

**Lemma 7.2.** *Let  $-\infty < a < c < l < d < b < \infty$ , and suppose that the function  $f_1$  is concave on  $[a, b]$  and not affine on  $[c, l]$ , and has a derivative at points  $c, l$ , and  $d$ . Then there exists a number  $k > 0$  such that  $|G_i(c, x)| > k$  for all  $x \in [l, d] \cap C_i$ , where  $G_i$  and  $C_i$  are defined as in Lemma 7.1.*

*Proof.* Since

$$G(c, x) = \frac{f'_i(c) - \frac{f_i(x) - f_i(c)}{x - c}}{f'_i(x) - \frac{f_i(x) - f_i(c)}{x - c}},$$

we have only to prove that the inequalities

$$\left| f'_i(c) - \frac{f_i(x) - f_i(c)}{x - c} \right| > \left| f'_i(c) - \frac{f_i(l) - f_i(c)}{l - c} \right| \neq 0 \quad (7.8)$$

are valid, and that when  $h \in ]d, b[ \cap C_i$ , the inequalities

$$\left| f'_i(c) - \frac{f_i(x) - f_i(c)}{x - c} \right| < \left| f'_i(h) - \frac{f_i(h) - f_i(c)}{h - c} \right| \neq \infty \quad (7.9)$$

are valid too.

The validity of inequalities (7.8) follows from the concavity and nonaffineness of the functions  $f_i$  on the segment  $[c, l]$ , since under these conditions, as was already noted, the inequalities

$$\frac{f_i(x) - f_i(c)}{x - c} < \frac{f_i(l) - f_i(c)}{l - c} < f'_i(c)$$

are satisfied.

To prove (7.9) first note that if  $f'_i(x) = f'_i(h)$ , then

$$\frac{f_i(x) - f_i(c)}{x - c} > \frac{f_i(h) - f_i(c)}{h - c},$$

and since the concavity of  $f_i$  implies that

$$f_i'(x) \geq \frac{f_i(x) - f_i(c)}{x - c},$$

inequality (7.9) in this case is valid automatically. It remains only to consider the case  $f_i'(x) > f_i'(h)$ , since on account of the concavity of  $f_i$ , the reverse inequality is impossible.

Rewriting the inequality

$$f_i'(h) \leq \frac{f_i(h) - f_i(x)}{h - x}$$

in the form

$$f_i(x) \leq f_i'(h) + f_i'(h)(x - h),$$

we obtain

$$\begin{aligned} f_i(h) + f_i'(h)(c - h) - f_i(c) &= [f_i(h) + f_i'(h)(x - h)] - f_i(c) + f_i'(h)(c - x) \\ &\geq f_i(x) - f_i(c) + f_i'(h)(c - x) \\ &> f_i(x) - f_i(c) + f_i'(c)(c - x). \end{aligned}$$

But this means that

$$\begin{aligned} \frac{f_i(x) - f_i(c)}{x - c} - f_i'(x) &< \frac{f_i(h) - f_i(c)}{h - c} - f_i'(h) \frac{h - c}{x - c} \\ &= \frac{h - c}{x - c} \left[ \frac{f_i(h) - f_i(c)}{h - c} - f_i'(h) \right], \end{aligned}$$

and a fortiori,

$$\frac{f_i(x) - f_i(c)}{x - c} - f_i'(x) < \frac{f_i(h) - f_i(c)}{h - c} - f_i'(h),$$

since both parts of this inequality are nonnegative. And it is equivalent to (7.9). The lemma is proved.

(7) We have the following lemma.

**Lemma 7.3.** *Let  $(f_1, f_2) \in \mathfrak{L}([a, b])$ ,  $-\infty < a < c < d < b < +\infty$ ,  $f_1(c) = f_2(c)$ ,  $f_1(d) = f_2(d)$ ,  $c, d \in C_1 \cap C_2$ ,  $f_1'(c) = f_2'(c)$ . Then  $f_1(x) = f_2(x)$  for all  $x \in [a, b]$ .*

*Proof.* First we shall prove the required relationship for  $x \in ]a, b[$ . Suppose the contrary, i.e., that there exists  $z \in ]c, b[$  such that  $f_1(z) \neq f_2(z)$ , or without loss of generality,  $f_1(z) < f_2(z)$ .

If the function  $f_1$  is affine on  $[c, z]$ , then as we have already seen, the function  $f_2$  on  $[c, z]$  is also affine, and hence the condition

$$f_1'(c) = f_2'(c)$$

contradicts

$$f_1(z) \neq f_2(z).$$

Thus, we are interested only in the case in which  $f_1$  and  $f_2$  are not affine on  $[c, z]$ .

Let  $z < d$ . We set

$$m = \inf\{x \in [z, d] : f_1(x) = f_2(x)\}.$$

By virtue of the continuity of the concave functions  $f_1$  and  $f_2$  on  $]a, b[$ , it is clear that

$$f_1(m) = f_2(m).$$

Our goal now is to prove that with regard to the properties of the functions  $f_1$  and  $f_2$ , the inequality

$$f_1(z) < f_2(z)$$

implies the inequality

$$f_1(m) < f_2(m),$$

and thus the first is incorrect. To this end we shall proceed in the following way. Represent each of the functions  $f_i$ ,  $i = 1, 2$ , as a sum of two continuous monotonic functions

$$f_i = f_{i1} + f_{i2}.$$

The possibility of such representation follows from a remark in [5], where the following fact is asserted: if a function is concave on  $]a, b[$ , then either it is strictly monotonic on  $]a, b[$  or there exists a point  $u \in ]a, b[$  such that  $f$  is strictly increasing to the left of this point and strictly decreasing to the right of it, or finally, that there exist two points  $u, v \in ]a, b[$  such that  $u < v$  and the function  $f$  is strictly increasing on  $]a, b[$ , constant on  $]u, v[$ , and strictly decreasing on  $]v, b[$ . It is clear that setting  $u = v = b$  in the first of these cases and  $u = v$  in the second one, we obtain the required representation on introducing the functions  $f_{i1}$  and  $f_{i2}$  by the formulas

$$f_{i1}(x) = \begin{cases} f_i(u) - f_i(u) & \text{if } x \in ]a, u[, \\ 0 & \text{if } x \in ]u, b[; \end{cases}$$

and

$$f_{i2}(x) = \begin{cases} f_i(u) & \text{if } x \in ]a, v[, \\ f_i(x) & \text{if } x \in ]v, b[. \end{cases}$$

Now making use of the fact that a continuous monotonic function is clearly absolutely continuous, we conclude that the functions  $f_i, i = 1, 2$ , are also absolutely continuous.

Take an arbitrary point  $q \in ]z, m[$ . It is obvious that the function

$$\delta : [z, q] \rightarrow \mathbb{R}, \quad \delta(x) = f_2(x) - f_1(x)$$

is absolutely continuous, does not vanish at any point, and is positive at  $x = z$ . Therefore  $K_1, K_2 > 0$  may be found such that

$$K_1 \leq \delta(x) \leq K_2$$

for any  $x \in [z, m]$ . From here it follows that the function  $f(x) = \ln(\delta(x))$  is also absolutely continuous at  $x \in [z, m]$  and so can be reconstructed from its derivative.

Indeed, it is known [67] that given an absolutely continuous function  $\delta$  on the segment  $[z, m]$  with values in the segment  $[K_1, K_2]$ , and given as well a function  $g$  (in our case  $\ln(\cdot)$ ) satisfying a Lipschitz condition, then the composite function  $g(\delta(\cdot))$  will also be absolutely continuous.

Thus the function  $f = \ln(\delta(\cdot))$  can be reconstructed on  $[z, m]$  by means of its derivative (defined on almost all  $x \in [z, m]$ ), i.e.,

$$f(q) = f(z) + \int_z^q f'(x)dx = f(z) + \int_z^q \frac{f'_2(x) - f'_1(x)}{f_2(x) - f_1(x)} dx.$$

It will be shown below that

$$f'_2(x) - f'_1(x) = [f_2(x) - f_1(x)] \frac{1 - [G_i(c, x)]^{-1}}{x - c} \tag{7.10}$$

for all  $x \in C_1 \cap C_2 \cap [z, m], i = 1, 2$ .

Therefore, making use of Lemma 7.2, one can write

$$\begin{aligned} \left| \int_z^q \frac{f'_2(x) - f'_1(x)}{f_2(x) - f_1(x)} dx \right| &= \left| \int_z^q \frac{1 - [G_i(c, x)]^{-1}}{x - c} dx \right| \\ &< \int_z^q \frac{K^{-1} + 1}{z - c} dx \leq (m - z)(K^{-1} + 1)(z - c)^{-1}, \end{aligned}$$

from which it follows that

$$|f(q)| \leq (m-z)(K^{-1}+1)(z-c)^{-1} + f(z) \stackrel{\text{def}}{=} K_0.$$

Therefore,

$$|\ln(\delta(q))| \leq K_0 \quad \text{for all } q \in ]z, m[,$$

or equivalently,

$$|f_2(q) - f_1(q)| \geq e^{-K_0} > 0 \quad \text{for all } q \in ]z, m[,$$

from which by virtue of continuity we have

$$f_2(m) \neq f_1(m),$$

contrary to the choice of  $m$ . It remains to prove only (7.10).

Since

$$f_1(c) = f_2(c), \quad f'_1(c) = f'_2(c),$$

we have according to Lemma 7.1,

$$\frac{f'_i(c) - \frac{f_1(x) - f_i(c)}{x-c}}{f'_1(x) - \frac{f_1(x) - f_i(c)}{x-c}} = \frac{f'_i(c) - \frac{f_2(x) - f_i(c)}{x-c}}{f'_2(x) - \frac{f_2(x) - f_i(c)}{x-c}}$$

for all  $x \in C_1 \cap C_2 \cap ]z, m]$ ,  $i = 1, 2$ . This implies

$$\frac{f_i(c) - f_1(x) + f'_i(c)(x-c)}{f_i(c) - f_1(x) + f'_1(x)(x-c)} = \frac{f_i(c) - f_2(x) + f'_i(c)(x-c)}{f_i(c) - f_2(x) + f'_2(x)(x-c)},$$

and after reducing to a common denominator, collecting similar terms, and canceling the common factor  $(x-c) > 0$ , we obtain

$$\begin{aligned} & -f'_i(c)f_2(x) + f_i(c)f'_2(x) - f_1(x)f'_2(x) + f'_i(c)f'_2(x)(x-c) \\ & = -f'_i(c)f_1(x) + f_i(c)f'_1(x) - f_2(x)f'_1(x) + f'_i(c)f'_1(x)(x-c). \end{aligned}$$

Next,

$$f'_2(x) = f'_1(x) \frac{f_i(c) - f_2(x) + f'_i(c)(x-c)}{f_i(c) - f_1(x) + f'_i(c)(x-c)} + \frac{f'_i(c)[f_2(x) - f_1(x)]}{f_i(c) - f_1(x) + f'_i(c)(x-c)},$$

and finally,

$$f'_2(x) - f'_1(x) = \frac{[f_2(x) - f_1(x)][f'_i(c) - f'_1(x)]}{f_i(c) - f_1(x) + f'_i(c)(x-c)}.$$

At the same time,

$$1 - [G_i(c, x)]^{-1} = 1 - \frac{f_1'(x) - \frac{f_1(x) - f_1(c)}{x - c}}{f_i'(c) - \frac{f_1(x) - f_1(c)}{x - c}} = \frac{f_i'(c) - f_1'(x)}{f_i'(c) - \frac{f_1(x) - f_1(c)}{x - c}},$$

i.e.,

$$\frac{1 - [G_i(c, x)]^{-1}}{x - c} = \frac{f_i'(c) - f_1'(x)}{f_i'(c) - f_1'(x) + f_i'(c)(x - c)}.$$

The relationship (7.10) is proved similarly, and this proves the assertion of Lemma 7.3 for the case  $x \in [c, b]$ ,  $z > d$ .

Now let  $z > d$ . In this case the reasoning is similar, with the only difference being that now by  $m$  we understand

$$\sup \{x \in [d, z] : f_1(x) = f_2(x)\}$$

and choose the point  $q$  from the half-interval  $]m, z]$ . This completes the proof for the case  $x \in [c, b]$ .

It remains only to extend it to  $x \in [a, c]$ . In fact, we shall carry out a proof suitable for all  $x \in [a, b]$ . To this end, suppose

$$\begin{aligned} \xi &: [a, b] \rightarrow [a, b], & \xi(x) &= a + b - x, \\ g_i &: [a, b] \rightarrow \mathbb{R}, & g_i(x) &= f_i(\xi(x)) \quad (i = 1, 2), \\ & & \tilde{c} &= \xi(d), \quad \tilde{d} = \xi(c). \end{aligned}$$

For functions  $g_1, g_2$  thus defined and points  $\tilde{c}, \tilde{d}$ , all the conditions of Lemma 7.3 are satisfied, and so by what we have proved above,

$$g_1(x) = g_2(x) \quad \text{for all } x \in [\tilde{c}, b],$$

or taking into account that  $x = a + b - \xi$  when  $\xi = a + b - x$ , we obtain

$$f_1(\xi) = f_2(\xi) \quad \forall \xi \in [a, d].$$

This relation differs from the desired relation in notation only. The lemma is proved.

(8) We have the following lemma.

**Lemma 7.4.** *Let  $f, g$ , and  $h$  be three real functions defined on the segment  $[a, b] \subset \mathbb{R}$ . The functions  $f$  and  $g$  are not affine on  $[a, b]$ , we have  $(f, g) \in \mathcal{M}([a, b])$ , and there exist  $\lambda, \alpha, \beta \in \mathbb{R}$  such that*

$$h : [a, b] \rightarrow \mathbb{R}, \quad h(x) = \lambda g(x) + \alpha x + \beta,$$

and

$$(f, h) \in \mathfrak{U}([a, b]) \Leftrightarrow \lambda > 0.$$

*Proof.* Let

$$\begin{aligned} x_1, x_2, \dots, x_m, x'_1, x'_2, \dots, x'_n &\in [a, b] \\ \alpha_1, \alpha_2, \dots, \alpha_m, \alpha'_1, \alpha'_2, \dots, \alpha'_n &\in [0, 1], \\ \sum_{i=1}^m \alpha_i &= \sum_{j=1}^n \alpha'_j = 1, \quad \sum_{i=1}^m \alpha_i x_i = \sum_{j=1}^n \alpha'_j x'_j. \end{aligned}$$

Then

$$\begin{aligned} &\operatorname{sgn} \left[ \sum_{i=1}^m \alpha_i h(x_i) - \sum_{j=1}^n \alpha'_j h(x'_j) \right] \\ &= \operatorname{sgn} \left[ \sum_{i=1}^m \alpha_i (\lambda g(x_i) + \alpha x_i + \beta) - \sum_{j=1}^n \alpha'_j (\lambda g(x'_j) + \alpha x_j + \beta) \right] \\ &= \operatorname{sgn} \left[ \lambda \left[ \sum_{i=1}^m \alpha_i g(x_i) - \sum_{j=1}^n \alpha'_j g(x'_j) \right] \right] \\ &= \operatorname{sgn}(\lambda) \operatorname{sgn} \left[ \sum_{i=1}^m \alpha_i f(x_i) - \sum_{j=1}^n \alpha'_j f(x'_j) \right], \end{aligned}$$

and the statement of the lemma follows immediately.

We now introduce one more notation that will be useful in what follows. If  $x_1, x_2, \dots, x_n \in E$  are linearly independent vectors of a linear space  $E$ , and  $f$  is an arbitrary real function defined on  $E$  or on a subset containing the indicated collection of points, then one can speak about the affine function  $g$  defined on the linear span of vectors  $x_1, x_2, \dots, x_n$  and taking values  $f(x)$  at the points  $x_i$ ,  $1 \leq i \leq n$ . Clearly this function is well defined and can be given by the formula

$$g(x) = \sum_{i=1}^n \alpha_i f(x_i) \quad \text{when } x = \sum_{i=1}^n \alpha_i x_i, \quad \alpha_i \in \mathbb{R}.$$

Now, making use of this formula, we can consider a new function, which we shall denote by  $[f_{x_1, x_2, \dots, x_n}]$  and define it in the following way. If  $A$  is the intersection of the domain of definition of  $f$  and of the linear span of the family of vectors  $x_1, x_2, \dots, x_n$ , then

$$[f_{x_1, x_2, \dots, x_n}] : A \rightarrow \mathbb{R}, \quad [f_{x_1, x_2, \dots, x_n}](x) = f(x) - g(x).$$

This notation allows to formulate the following corollary.

**Corollary 7.1.**

$$(f, g) \in \mathfrak{U}([a, b]) \Leftrightarrow ([f_{a,b}], [g_{a,b}]) \in \mathfrak{U}([a, b]).$$

*Proof.* Clearly, if even one of the four functions under consideration is affine on  $[a, b]$ , then the rest are affine too, so any pair of them is contained in  $\mathfrak{U}([a, b])$ . If none of them is affine, then in order to make use of Lemma 7.4, it is sufficient to note that

$$\begin{aligned} [f_{a,b}](x) &= f(x) - \left[ \frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b) \right] \\ &= 1 \cdot f(x) + \frac{f(a) + f(b)}{b-a} x + \frac{bf(a) + af(b)}{a-b}. \end{aligned}$$

(9) We have the following lemma.

**Lemma 7.5.** *If  $(f, g) \in \mathfrak{u}([a, b])$ , then there exists  $\lambda > 0$  such that*

$$[f_{a,b}](x) = \lambda [g_{a,b}](x) \quad \text{for all } x \in [a, b].$$

*Proof.* First note that even if one of the functions  $f$  and  $g$  is affine on  $[a, b]$ , we can take as  $\lambda$  any positive number. Therefore only that case will be of interest to us when both these functions are not affine on  $[a, b]$ . In this case, in view of the almost everywhere differentiability of a concave function defined on a segment, one can, as we have seen, choose numbers  $c, d \in ]a, b[$  such that the functions  $f$  and  $g$  are not affine on  $[c, d]$  and are differentiable at points  $c$  and  $d$ . Consider the system of linear equations with unknowns  $\lambda, \alpha, \beta \in \mathbb{R}$ , where for convenience instead of  $[f_{a,b}]$  and  $[g_{a,b}]$  we write  $f^*$  and  $g^*$ :

$$\begin{aligned} f^*(c) &= \lambda g^*(c) + \alpha c + \beta, \\ f^*(d) &= \lambda g^*(d) + \alpha d + \beta, \\ (f^*)'(c) &= \lambda (g^*)'(c) + \alpha. \end{aligned}$$

One can easily show that the determinant of this system is given by

$$\begin{aligned} \Delta &= \begin{vmatrix} g^*(c) & c & 1 \\ g^*(d) & d & 1 \\ (g^*)'(c) & 1 & 0 \end{vmatrix} = -g^*(c) + c(g^*)'(c) + g^*(d) - d(g^*)'(c) \\ &= g^*(d) - [g^*(c) + (d-c)(g^*)'(c)] < 0, \end{aligned}$$

and therefore  $\Delta < 0$ , since the function  $g^*$  together with  $g$  is concave and not affine on  $[c, d]$ .

Thus, the system has a solution, which we denote by  $(\lambda^*, \alpha^*, \beta^*)$ . In addition,

$$\lambda^* = \frac{f^*(d) - [f^*(c) + (d-c)(f^*)'(c)]}{g^*(d) - [g^*(c) + (d-c)(g^*)'(c)]} > 0,$$

and we consider the function

$$h : [a, b] \rightarrow \mathbb{R}, \quad h(x) = \lambda^* g^*(x) + \alpha^* x + \beta^*.$$

Then according to Lemma 7.4 we have  $(h, g^*) \in \mathfrak{U}([a, b])$ , and according to the corollary of that lemma, by virtue of transitivity of the relation  $\mathfrak{U}([a, b])$ , we obtain  $(h, f^*) \in \mathfrak{U}([a, b])$ .

Further, in view of

$$h(c) = f^*(c), \quad h(d) = f^*(d), \quad h'(c) = (f^*)'(c),$$

on the basis of Lemma 7.3 we can conclude that

$$h(x) = f^*(x) \quad \text{for all } x \in [a, b]. \quad (7.11)$$

It remains to prove that  $\alpha^* = \beta^* = 0$ . To this end, note that

$$f^*(a) = f^*(b) = g^*(a) = g^*(b) = 0,$$

and so from (7.11), it follows that

$$\begin{aligned} 0 &= f^*(a) = \lambda^* g^*(a) + \alpha^* a + \beta^* = \alpha^* a + \beta^*, \\ 0 &= f^*(b) = \lambda^* g^*(b) + \alpha^* b + \beta^* = \alpha^* b + \beta^*, \end{aligned}$$

whence

$$\alpha^* = \beta^* = 0,$$

i.e.,

$$h(x) = \lambda^* g^*(x) = f^*(x) \quad \text{for all } x \in [a, b].$$

**Corollary 7.2.** *If  $e_1, e_2 \in E$ ,  $e_1 \neq e_2$ ,*

$$[e_1, e_2] = \{x \in E : \exists \alpha \in [0, 1], (x = (1 - \alpha)e_1 + \alpha e_2)\}, \quad (f, g) \in \mathfrak{U}([e_1, e_2]),$$

*then there exists  $\lambda > 0$  such that*

$$[f_{e_1, e_2}](x) = \lambda [g_{e_1, e_2}](x), \quad \forall x \in [e_1, e_2].$$

*Proof.* Let us parameterize the segment  $[e_1, e_2]$  in the form

$$\varphi : [0, 1] \rightarrow E, \quad \varphi(t) = (1 - t)e_1 + te_2$$

and set

$$\tilde{f}(t) = f(\varphi(t)), \quad \tilde{g}(t) = g(\varphi(t)) \quad (t \in [0, 1]).$$

It is easy to show that  $(\tilde{f}, \tilde{g}) \in \mathfrak{U}([0, 1])$ , and thus, by Lemma 7.5,

$$[\tilde{f}_{0,1}] = \lambda [\tilde{g}_{0,1}]$$

at some  $\lambda > 0$ .

But

$$\begin{aligned} [\tilde{f}_{0,1}](t) &= \tilde{f}(t) - [(1-t)\tilde{f}(0) + t\tilde{f}(1)] = f(\varphi(t)) - [(1-t)f(e_1) + tf(e_2)] \\ &= f(\varphi) - \left[ \frac{e_2 - \varphi}{e_2 - e_1} f(e_1) + \frac{\varphi - e_1}{e_2 - e_1} f(e_2) \right] = [f_{e_1, e_2}](\varphi), \end{aligned}$$

where  $\varphi = (1-t)e_1 + te_2$ .

Similarly,

$$[\tilde{g}_{0,1}](t) = [g_{e_1, e_2}](\varphi),$$

and so

$$[f_{e_1, e_2}](x) = \lambda [g_{e_1, e_2}](x) \text{ for all } x \in [e_1, e_2].$$

(10) We have the following lemma.

**Lemma 7.6.** *Let  $A \subset E$  be some convex set,  $(f, g) \in \mathfrak{U}(A)$ ,  $e_1, e_2, e_3, e_4 \in A$ . Then there exists  $\lambda > 0$  such that*

$$[f_{e_i, e_j}] = \lambda [g_{e_i, e_j}] \quad \text{for any } i, j \in \{1, 2, 3, 4\}.$$

*Proof.* Without loss of generality we may assume that none of the functions  $f|_{[e_i, e_j]}$  and  $g|_{[e_i, e_j]}$  for  $i, j \in \{1, 2, 3, 4\}$ ,  $i \neq j$  is affine, since if, for example,  $f|_{[e_i, e_j]}$  is affine, then clearly

$$[f_{e_1, e_2}](x) = [g_{e_1, e_2}](x) = 0 \quad \text{for all } x \in [e_1, e_2],$$

and if for all other pairs  $(i, j)$  the required  $\lambda$  exists, then it is suitable for this pair as well.

According to the corollary to Lemma 7.5,

$$[f_{e_1, e_3}] = \lambda_1 [g_{e_1, e_3}], \quad [f_{e_2, e_4}] = \lambda [g_{e_2, e_4}],$$

and therefore if we prove that  $\lambda_1 = \lambda$ , then the validity of the lemma will follow by virtue of the arbitrariness of the choice of pairs  $(e_1, e_3)$  and  $(e_2, e_4)$  as well as the nonaffineness of the functions  $f|_{[e_1, e_3]}$  and  $f|_{[e_2, e_4]}$ .

The idea of the proof consists in the following. As will be shown below, one can select the points  $x_1, x_3 \in [e_1, e_3]$  and  $x_2, x_4 \in [e_2, e_4]$ , and also the real

numbers  $\delta_0 > 0$  and  $r$ , such that for all  $\delta \in ]0, \delta_0[$ ,

$$\begin{aligned}x'_1 &= x_1 + \delta(e_3 - e_1), & x'_3 &= x_3 - \delta(e_3 - e_1), \\x'_2 &= x_2 + r\delta(e_4 - e_2), & x'_4 &= x_4 - r\delta(e_4 - e_2).\end{aligned}$$

From  $\lambda_1 \neq \lambda_2$  it follows that

$$\operatorname{sgn} \sum_{i=1}^4 [f(x_i) - f(x'_i)] \neq \operatorname{sgn} \sum_{i=1}^4 [g(x_i) - g(x'_i)],$$

in spite of the obvious relationship

$$\sum_{i=1}^4 \frac{1}{4} x_i = \sum_{i=1}^4 \frac{1}{4} x'_i$$

and the condition  $(f, g) \in \mathfrak{U}(A)$ .

Precisely, this proof can be carried out as follows. We parameterize the segments  $[e_1, e_3]$  and  $[e_2, e_4]$ :

$$\begin{aligned}\varphi_1 : [0, 1] &\rightarrow E, & \varphi_1(t) &= (1-t)e_1 + te_3, \\ \varphi_2 : [0, 1] &\rightarrow E, & \varphi_2(t) &= (1-t)e_2 + te_4.\end{aligned}$$

Obviously, the functions  $f_i$  and  $g_i$  are concave and not affine on  $[0, 1]$ , and

$$(f_i, g_i) \in \mathfrak{U}([0, 1]) \quad (i = 1, 2).$$

Therefore there exist  $t_1, t_2, t_3, t_4 \in ]0, 1[$  such that the functions  $f_i$  and  $g_i$ ,  $i = 1, 2$ , are differentiable at all these points and we have the relations

$$f'_1(t_1) \neq f'_1(t_3), \quad f'_2(t_2) = f'_2(t_4), \quad (7.12)$$

$$g'_1(t_1) \neq g'_1(t_3), \quad g'_2(t_2) \neq g'_2(t_4). \quad (7.13)$$

Let

$$\begin{aligned}\delta_1, \delta_2 &> 0, \\ x_i &= \begin{cases} \varphi_1(t_i), & \text{if } i = 1, 3, \\ \varphi_2(t_i), & \text{if } i = 2, 4, \end{cases} \\ t'_1 &= t_1 + \delta_1, & t'_3 &= t_3 - \delta_1, & t'_2 &= t_2 + \delta_2, \\ t'_4 &= t_4 - \delta_2, & x'_i &= \begin{cases} \varphi_1(t'_i), & \text{if } i = 1, 3; \\ \varphi_2(t'_i), & \text{if } i = 2, 4. \end{cases}\end{aligned}$$

Then

$$\begin{aligned}x'_1 &= x_1 + \delta_1(e_3 - e_1), & x'_3 &= x_3 - \delta_1(e_3 - e_1), \\x'_2 &= x_2 + \delta_2(e_4 - e_2), & x'_4 &= x_4 - \delta_2(e_4 - e_2),\end{aligned}$$

and when  $\delta_1, \delta_2 \rightarrow 0$ , we have

$$\begin{aligned}\sum_{i=1}^4 [f(x'_i) - f(x_i)] &= (f_1(t'_1) - f_1(t_1)) + (f_1(t'_3) - f_1(t_3)) + (f_2(t'_2) - f_2(t_2)) + (f_2(t'_4) - f_2(t_4)) \\&= [f'_1(t_1) - f'_1(t_3)]\delta_1 + O(\delta_1) + [f'_2(t_2) - f'_2(t_4)]\delta_2 + O(\delta_2),\end{aligned}$$

and analogously

$$\sum_{i=1}^4 [g(x'_i) - g(x_i)] = [g'_1(t_1) - g'_1(t_3)]\delta_1 + O(\delta_1) + [g'_2(t_2) - g'_2(t_4)]\delta_2 + O(\delta_2).$$

Now note that

$$f'_1(t_1) - f'_1(t_3) = [(f_1)_{0,1}]'(t_1) - [(f_1)_{0,1}]'(t_3).$$

Similar equations are valid for the functions  $f_2, g_1, g_2$ . Therefore, from

$$[(f_i)_{0,1}] = \lambda_i [(g_i)_{0,1}], \quad \lambda_1 \neq \lambda_2,$$

it follows that

$$\frac{f'_1(t_1) - f'_1(t_3)}{g'_1(t_1) - g'_1(t_3)} \neq \frac{f'_2(t_2) - f'_2(t_4)}{g'_2(t_2) - g'_2(t_4)},$$

and thus there exists  $r \neq 0$  such that

$$[[f'_1(t_1) - f'_1(t_3)] + r[f'_2(t_2) - f'_2(t_4)]] [[g'_1(t_1) - g'_1(t_3)] + r[g'_2(t_2) - g'_2(t_4)]] < 0.$$

With regard to (7.12) and (7.13), this means that there exists  $\delta_0 > 0$  such that for all  $\delta_1 \in ]0, \delta_0[$ ,  $\delta_2 = r\delta_1$ , the following chain of relations holds:

$$\begin{aligned}\operatorname{sgn} \sum_{i=1}^4 [f(x'_i) - f(x_i)] &= \operatorname{sgn} [(f'_1(t_1) - f'_1(t_3)) + r(f'_2(t_2) - f'_2(t_4))] \\&\neq \operatorname{sgn} [(g'_1(t_1) - g'_1(t_3)) + r(g'_2(t_2) - g'_2(t_4))] \\&= \operatorname{sgn} \sum_{i=1}^4 [g(x'_i) - g(x_i)].\end{aligned}$$

From this follows the validity of the statement of Lemma 7.6, as was shown above.

(11) We have the following lemma.

**Lemma 7.7.** *Let  $e_1, e_2, \dots, e_n$  be a collection of linearly independent vectors belonging to some convex set  $A \subset E$ . Let further*

$$(f, g) \in \mathfrak{U}(A), \quad f(e_i) = g(e_i) = 0$$

for  $i = 1, 2, \dots, n$ . Then there exists  $\lambda > 0$  such that  $f(x) = \lambda g(x)$  for all  $x$  from the convex hull of the vectors  $e_1, e_2, \dots, e_n$ .

*Proof.* We will carry out the proof by induction on  $n$ . For  $n = 1$  the lemma's statement follows immediately from the corollary of Lemma 7.5. Suppose that this statement is valid for all  $n < k$ . In this case, denoting the convex hull of the vectors  $e_1, e_2, \dots, e_k$  by  $M$  and the convex hull of vectors  $e_1, e_2, \dots, e_k$  by  $M_1$ , we have by induction that

$$f|_{M_1} = \lambda g|_{M_1}.$$

We set  $F = \{x \in M_1 : g(x) \neq 0\}$  and assign to any point  $x \in M$  a point  $y_x \in M_1$  such that  $x \in [e_1, y_x]$ . We now consider two cases, depending on whether the set  $F$  is empty.

Case 1. Let  $F = \emptyset$ . Then for any  $x \in M$  we have

$$f|_{[e_1, y_x]} = [f_{e_1, y_x}], \quad g|_{[e_1, y_x]} = [g_{e_1, y_x}],$$

and if there exists a point  $x_0$  at which  $g(x_0) \neq 0$ , then writing  $\lambda = \frac{f(x_0)}{g(x_0)}$ , by the corollary of Lemma 7.5, we obtain

$$f(x) = \lambda g(x) \quad \text{for all } x \in [e_1, y_{x_0}],$$

and if we now take an arbitrary point  $x \in M$  and apply Lemma 7.6 to the points  $e_1, y_{x_0}, e_1, y_x$ , we obtain

$$f(x) = \lambda g(x) \quad \text{for all } x \in M.$$

If the function  $g$  is identically equal to zero on  $M$ , then it is affine on  $M$ , and so the function  $f$  is affine too, being necessarily identical to zero in this case, so that we can take as  $\lambda$  any positive number.

Case 2. Now let  $F \neq \emptyset$ . Choose an arbitrary  $y \in F$  and apply Lemma 7.6 to the points  $e_1, e_2, y, y$ . We obtain

$$[f_{e_1, y}] = \lambda_1 [g_{e_1, y}],$$

where  $\lambda_1$  is a positive number, existing by the induction hypothesis, such that

$$f|M = \lambda_1 g|M_1.$$

Now taking arbitrary  $x \in M$  and applying Lemma 7.6 to the points  $e_1, y_1, e_1, y_x$ , we obtain

$$[f_{e_1, y_x}] = \lambda_1 [g_{e_1, y_x}],$$

i.e., for any  $x \in M$  we have the chain of equalities

$$\begin{aligned} [f_{e_1, y_x}](x) &= f(x) - \left[ \frac{y_x - x}{y_x - e_1} f(e_1) + \frac{x - e_1}{y_x - e_1} f(y_x) \right] = f(x) - \frac{x - e_1}{y_x - e_1} f(y_x) \\ &= f(x) - \lambda_1 \frac{x - e_1}{y_x - e_1} g(y_x) = \lambda_1 [g_{e_1, y_x}] x = \lambda_1 \left[ g(x) - \frac{x - e_1}{y_x - e_1} g(y_x) \right], \end{aligned}$$

whence one can see immediately that

$$f(x) = \lambda_1 g(x) \quad \text{for all } x \in M.$$

Lemma 7.7 is proved.

(12) The following lemma is of prime importance in the proof of Theorem 7.1.

**Lemma 7.8.** *Let  $P_0, P_1, P_2, \dots, P_n \in \mathcal{P}(\Theta), \alpha_1, \alpha_2, \dots, \alpha_n \in ]0, 1[$ ,*

$$\begin{aligned} \sum_{i=1}^n \alpha_i &= 1, \quad P_0 = \sum_{i=1}^n \alpha_i P_i, \\ \text{i.e., } P_0(\theta) &= \sum_{i=1}^n \alpha_i P_i(\theta) \quad \forall \theta \in \Theta. \end{aligned}$$

*Then there exists a stochastic experiment  $\eta = (W, \mu)$  on  $\Theta$  such that the set  $W$  consists of  $n$  points (denote them by  $w_1, w_2, \dots, w_n$ ) and*

$$P_0(\theta/w_i) = P_i(\theta) \quad \text{for all } i = 1, 2, \dots, n, \quad \theta \in \Theta.$$

*Proof.* Suppose

$$\mu(w_i/\theta) = \begin{cases} \frac{\alpha_i P_i(\theta)}{P_0(\theta)}, & \text{if } P_0(\theta) \neq 0, \\ \frac{1}{n}, & \text{if } P_0(\theta) = 0. \end{cases}$$

Since

$$\mu(w_i/\theta) \in [0, 1]$$

and

$$\sum_{i=1}^n \mu(w_i/\theta) = \begin{cases} \frac{\sum_{i=1}^n \alpha_i P_i(\theta)}{P_0(\theta)} = 1 & \text{for } P_0(\theta) \neq 0, \\ n \frac{1}{n} = 1 & \text{for } P_0(\theta) = 0, \end{cases}$$

it is clear that  $\mu(\cdot/\cdot)$  is the transition probability from  $\Theta$  to  $W$ , and hence  $\eta$  is a stochastic experiment. It remains to prove that this is the one we are looking for.

Set  $\Theta_0 = \{\theta \in \Theta : P_0(\theta) \neq 0\}$ . If  $\theta \in \Theta_0$ , then

$$\begin{aligned} P_0(\theta/w_i) &= \frac{\mu(w_i/\theta)P_0(\theta)}{\sum_{\theta' \in \Theta} \mu(w_i/\theta')P_0(\theta')} \\ &= \frac{\frac{\alpha_i P_i(\theta)}{P_0(\theta)} P_0(\theta)}{\sum_{\theta' \in \Theta} \frac{\alpha_i P_i(\theta')}{P_0(\theta')} P_0(\theta') + \sum_{\theta' \in \Theta \setminus \Theta_0} \frac{1}{n} \cdot 0} \\ &= \frac{\alpha_i P_i(\theta)}{\alpha_i} = P_i(\theta). \end{aligned}$$

If  $\theta \in \Theta \setminus \Theta_0$ , then

$$P_0(\theta/w_i) = \frac{\frac{1}{n} P_0(\theta)}{\sum_{\theta' \in \Theta} \mu(w_i/\theta') P_0(\theta')} = \frac{0}{\alpha_i} = 0.$$

But  $P_i(\theta) = 0$  as well, since

$$P_0(\theta) = \sum_{i=1}^n \alpha_i P_i(\theta) = 0$$

and all  $\alpha_i$  differ from 0. Thus for all cases,  $P_0(\theta/w_i) = P_i(\theta)$ . The lemma is proved.

(13) We now resume the proof of Theorem 7.1. Let  $E = (\Theta \rightarrow \mathbb{R})$  be a linear space of real functions defined on  $\Theta$ . Obviously, the set  $\mathcal{P}_\Theta$  is a set of linearly independent vectors from the set  $E$ , and  $\mathcal{P}$  is their convex hull. Making use of Lemma 7.8, one can easily see that condition (1) in Definition 7.1 defining  $\mathfrak{N}(Z)$  can be rewritten in the form

$$(n, \gamma_Z) \in \mathfrak{U}(\mathcal{P}(\Theta)),$$

and so, taking into account condition (2), we obtain due to Lemma 7.7 that if  $n \in \mathfrak{N}(Z)$ , then there exists  $\lambda > 0$  such that

$$nP = \lambda n_Z(P) \quad \text{for all } P \in \mathcal{P}(\Theta).$$

The converse statement is obvious. Theorem 7.1 is proved.

**Corollary 7.3.** *The function  $n : \mathcal{P}(\Theta) \rightarrow \mathbb{R}$  satisfies condition (1) in the definition of the class  $\mathfrak{N}(Z)$  (Definition 7.1) if and only if there exists a positive number  $\lambda$  such that the function  $(n - \lambda n_Z)$  is affine on  $\mathcal{P}(\Theta)$ .*

*Proof.* As is seen from the proof of Theorem 7.1, condition (1) is equivalent to

$$(n, \gamma_Z) \in \mathfrak{U}(\mathcal{P}(\Theta)).$$

Therefore, writing

$$[n_{\mathcal{P}_\Theta}] = [n_{P_{\theta_1}, P_{\theta_2}, \dots, P_{\theta_k}}], \quad \text{where } \{\theta_1, \theta_2, \dots, \theta_k\} = \Theta,$$

and with  $P_\theta$  a distribution concentrated at the point  $\theta$ , we have from Lemma 7.7 that

$$[n_{\mathcal{P}_\Theta}] = \lambda n_Z, \quad \lambda > 0,$$

whence

$$n - \lambda n_Z = n - [n_{\mathcal{P}_\Theta}],$$

i.e., by the definition of the function  $[n_{\mathcal{P}_\Theta}]$  the function  $(n - \lambda n_Z)$  is affine on  $\mathcal{P}(\Theta)$ .

Conversely, let the function  $(n - \lambda n_Z)$  be affine on  $\mathcal{P}(\Theta)$ . Then for each

$$\begin{aligned} P_1, P_2, \dots, P_m, P'_1, P'_2, \dots, P'_n &\in \mathcal{P}(\Theta), \\ \alpha_1, \alpha_2, \dots, \alpha_m, \alpha'_1, \alpha'_2, \dots, \alpha'_n &\in ]0, 1[, \\ \sum_{i=1}^m \alpha_i = \sum_{j=1}^n \alpha'_j = 1, \quad \sum_{i=1}^m P_i \alpha_i = \sum_{j=1}^n P'_j \alpha'_j \end{aligned}$$

we obtain

$$(n - \lambda n_Z) \left( \sum_{i=1}^m \alpha_i P_i \right) = \sum_{i=1}^m \alpha_i (n - \lambda n_Z)(P_i) = \sum_{i=1}^m \alpha_i n(P_i) - \sum_{i=1}^m \alpha_i \lambda n_Z(P_i),$$

which is equal to

$$(n - \lambda n_Z) \left( \sum_{j=1}^n \alpha'_j P'_j \right) = \sum_{j=1}^n \alpha'_j n(P'_j) - \sum_{j=1}^n \alpha'_j \lambda n_Z(P'_j),$$

whence

$$\sum_{i=1}^m \alpha_i n(P_i) - \sum_{j=1}^n \alpha'_j n(P'_j) = \sum_{i=1}^m \alpha_i \lambda n_Z(P_i) - \sum_{j=1}^n \alpha'_j \lambda n_Z(P'_j).$$

From this we conclude that

$$(\mathbf{n}, \lambda \mathbf{n}_Z) \in \mathfrak{L}(\mathcal{P}(\Theta)).$$

It remains to note that

$$(\gamma_Z, \lambda \mathbf{n}_Z) \in \mathfrak{L}(\mathcal{P}(\Theta))$$

also. So by virtue of the transitivity of the relationship  $\mathfrak{L}(\mathcal{P}(\Theta))$ , we obtain the relation

$$(\mathbf{n}, \gamma_Z) \in \mathfrak{L}(\mathcal{P}(\Theta)),$$

which is equivalent to Condition 1 of Definition 7.2.

**Corollary 7.4.** *If  $Z = (\Theta, U, L) \in \mathbb{Z}(\Theta)$  and the set  $\Theta$  is finite, then each function  $\mathbf{n}$  from  $\mathfrak{N}(Z)$  is nonnegative, concave, and continuous in the natural topology of the set  $\mathcal{P}(\Theta)$  (as a subset of the finite-dimensional linear space of all real functions on  $\Theta$ ).*

*Proof.* Since the function  $\mathbf{n}_Z$  is concave and takes the value 0 on the set  $\mathcal{P}(\Theta)$ , and  $\mathbf{n} = \lambda \mathbf{n}_Z$  at some  $\lambda > 0$ , it is sufficient to prove only the continuity of  $\mathbf{n}_Z$ . And moreover, due to the form of this function and due to the obvious fact that an affine function is automatically continuous, one need prove only the continuity of the function  $\gamma_Z$ . We introduce the norm

$$\|f\| = \max_{\theta \in \Theta} |f(\theta)|, \quad f \in (\Theta \rightarrow \mathbb{R}).$$

If

$$\delta > 0, \quad P_1, \quad P_2 \in \mathcal{P}(\Theta), \quad \|P_1 - P_2\| < \delta, \quad u_1 \in U$$

and

$$\sum_{\theta \in \Theta} L(\theta, u_1) P_1(\theta) \leq \gamma_Z(P_1) + \delta_1,$$

then

$$\begin{aligned} \gamma_Z(P_2) &\leq \sum_{\theta \in \Theta} L(\theta, u_1) P_2(\theta) \\ &\leq \sum_{\theta \in \Theta} L(\theta, u_1) P_1(\theta) + \delta \text{Card}(\Theta) \max_{\theta \in \Theta} L(\theta, u_1) \\ &\leq \gamma_Z(P_1) + \delta(1 + \text{card}(\Theta) \max_{\theta \in \Theta} L(\theta, u_1)). \end{aligned} \quad (7.14)$$

Now suppose that the function  $\gamma_Z$  has a break at some point  $P_1 \in \mathcal{P}(\Theta)$ , which means that one of the following two possibilities is realized:

- (1)  $\limsup_{P \rightarrow P_1} \gamma_Z(P) > \gamma_Z(P_1)$ ;  
 (2)  $\liminf_{P \rightarrow P_1} \gamma_Z(P) < \gamma_Z(P_1)$ .

In case (1), writing

$$\lim_{P \rightarrow P_1} \sup \gamma_Z(P) - \gamma_Z(P_1) = \varepsilon,$$

we obtain that for any  $\delta > 0$  there exists  $P_2 \in \mathcal{P}(\Theta)$  such that

$$\|P_2 - P_1\| < \delta \quad \text{and} \quad \gamma_Z(P_2) > \gamma_Z(P_1) + \frac{\varepsilon}{2},$$

but this contradicts the inequality (7.14) when

$$\delta = \frac{\varepsilon}{4(1 + \text{card}(\Theta) \max_{\theta \in \Theta} L(\theta, u_1))}.$$

In case (2), having set

$$\gamma_Z(P_1) - \lim_{P \rightarrow P_1} \inf \gamma_Z(P) = \varepsilon, \quad \min\{x > 0 : \exists \theta, P_1(\theta) = x\} = r,$$

we take  $P_2 \in \mathcal{P}(\Theta)$ , which exists by virtue of the definition of case (2), such that the inequalities

$$\gamma_Z(P_2) < \gamma_Z(P_1) - \frac{\varepsilon}{2}$$

and

$$\|P_2 - P_1\| < \frac{r\varepsilon}{2[\gamma_Z(P_1) - \inf L(\Theta, U)]}$$

are satisfied.

Let

$$t = \max\{x \in \mathbb{R} : P_1 + (P_2 - P_1)x \in \mathcal{P}(\Theta)\}, \quad P_3 = P_1 + (P_2 - P_1)t.$$

Clearly,

$$t \geq \frac{r}{\|P_2 - P_1\|}.$$

Then

$$P_2 = P_1 \left(1 - \frac{1}{t}\right) + P_3 \frac{1}{t},$$

and because of the concavity of the function  $\gamma_Z$ , we obtain

$$\gamma_Z(P_2) \geq \gamma_Z(P_1) \left(1 - \frac{1}{t}\right) + \gamma_Z(P_3) \frac{1}{t}.$$

This implies

$$\begin{aligned} \gamma_Z(P_3) &\leq t [\gamma_Z(P_2) - \gamma_Z(P_1)] + \gamma_Z(P_1) \leq \gamma_Z(P_1) + \frac{r}{\|P_2 - P_1\|} \times [\gamma_Z(P_2) - \gamma_Z(P_1)] \\ &< \gamma_Z(P_1) - \frac{r}{\|P_2 - P_1\|} \frac{\varepsilon}{2} < \gamma_Z(P_1) - \frac{r}{\frac{r\varepsilon}{2[\gamma_Z(P_1) - \inf L(\Theta, U)]}} \frac{\varepsilon}{2} = \inf L(\Theta, U). \end{aligned}$$

But this is impossible by the definition of the function  $\gamma_Z$ .

### 7.3 The Inverse Theorem

**Theorem 7.2.** *Let  $\Theta$  be a finite set, suppose the function  $\mathfrak{n} : \mathcal{P}(\Theta) \rightarrow \mathbb{R}$  is concave and continuous in the natural topology of the set  $\mathcal{P}(\Theta)$  and that  $\mathfrak{n}(P) = 0$  if  $P$  is concentrated at one point, i.e.,  $P \in \mathcal{P}_\Theta$ . Then there exists  $Z = (\Theta, U, L) \in \mathbb{Z}(\Theta)$  such that  $\mathfrak{n} = \mathfrak{n}_Z$ .*

*Proof.* Let the finite set  $\Theta$  contain  $n$  elements. Renumber them

$$\Theta = \{\theta_1, \dots, \theta_n\}$$

and map  $\mathcal{P}(\Theta)$  into  $\mathbb{R}^n$  by means of the function

$$T : P \mapsto (P(\theta_1), P(\theta_2), \dots, P(\theta_n)).$$

Obviously,  $T(\mathcal{P}(\Theta))$  turns out to be the  $(n-1)$ -dimensional simplex whose vertices are the points

$$x_1 = (1, 0, 0, \dots, 0), \quad x_2 = (0, 1, 0, \dots, 0), \quad \dots, \quad x_n = (0, 0, \dots, 1).$$

For brevity denote it by  $X$ . We introduce the function  $f : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$  given by the formula

$$f(x) = \begin{cases} -\mathfrak{n}(T^{-1}(x)), & \text{if } x \in X, \\ +\infty, & \text{if } x \notin X. \end{cases}$$

Clearly, it will be a convex eigenfunction with  $\text{dom } f = X$  (here and throughout the proof of this theorem we use notation and terminology from [72]).

Let  $U$  be a family of all affine functions  $u : \mathbb{R}^n \rightarrow \mathbb{R}$  satisfying the condition  $u(x) \leq f(x)$  for all  $x \in X$ . Clearly,  $U \neq \emptyset$  and

$$f(x) = \sup_{u \in U} u(x) \quad \forall x \in \mathbb{R}^n.$$

Now let

$$L : \Theta \times U \rightarrow \mathbb{R}, \quad L(\theta, u) = -u(T(P_\theta)),$$

where  $P_\theta$  is the distribution concentrated at the point  $\theta$ .

Our aim is to prove the two relations

$$\inf_{u \in U} \sum_{i=1}^n L(\theta_i, u) P(\theta_i) = n(P) \quad \forall P \in \mathcal{P}(\theta), \quad (7.15)$$

$$\inf_{u \in U} L(\theta, u) = 0 \quad \forall \theta \in \Theta. \quad (7.16)$$

In order to prove (7.15), note the following. If  $x \in X$ , then there exists one and only one collection of nonnegative numbers  $\lambda_i(x)$ ,  $i \in \overline{1, n}$ , such that

$$\sum_{i=1}^n \lambda_i(x) = 1 \quad \text{and} \quad \sum_{i=1}^n \lambda_i(x) x_i = x.$$

Furthermore, by virtue of the affineness of an arbitrary function  $u \in U$ , we have

$$u(x) = \sum_{i=1}^n \lambda_i(x) u(x_i).$$

Thus

$$\begin{aligned} n(P) &= -f(T(P)) = -\sup_{u \in U} u(T(P)) = -\sup_{u \in U} \left( \sum_{i=1}^n \lambda_i(T(P)) \times u(T(P_{\theta_i})) \right) \\ &= -\sup_{u \in U} \sum_{i=1}^n \lambda_i(T(P)) (-L(\theta_i, u)) = \inf_{u \in U} \sum_{i=1}^n L(\theta_i, u) \lambda_i(T(P)), \end{aligned}$$

and it remains only to note that

$$\lambda_i(T(P)) = P(\theta_i),$$

since

$$T(P) = \sum_{i=1}^n P(\theta_i) T(P_{\theta_i}).$$

Finally, in order to prove (7.16), we note that from the conditions of the theorem it follows that

$$\lim_{\substack{x \rightarrow x_j \\ x \in X}} f(x) = 0,$$

and therefore

$$\sup_{u \in U} u(x_i) \leq 0.$$

On the other hand, if we had strict inequality here, then there would exist a number  $r > 0$  such that the point  $(x_i, r)$  could not be separated by a hyperplane from the set  $\text{epi } f$ , contrary, for example, to [72, Theorem 2.1]. So

$$\sup_{u \in U} u(x_i) = \sup_{u \in U} (-L(\theta_i, u)) = - \inf_{u \in U} L(\theta_i, u) = 0$$

for all  $i \in \overline{1, n}$ .

Thus the equalities (7.15) and (7.16) are valid, whence the validity of Theorem 7.2 follows immediately.

**Theorem 7.3.** *Let*

$$P \in \mathcal{P}(\Theta), \quad \eta = (W, \mu), \quad \tilde{\Theta} = \Theta \times W,$$

$$q(A) = \sum_{(\theta, w) \in A} P(\theta) \mu(w/\theta) \text{ for all } A \subseteq \tilde{\Theta},$$

$$h: \tilde{\Theta} \rightarrow W, \quad h(\theta, w) = w, \quad S = (\Theta, U, L, \{P\}) \in \mathbf{S}(\Theta),$$

$$\tilde{L}: \tilde{\Theta} \times U \rightarrow \mathbb{R}, \quad \tilde{L}((\theta, w), u) = L(\theta, u), \quad \tilde{S} = (\tilde{\Theta}, U, \tilde{L}, \{q\}).$$

Then

$$\tilde{S} \in \mathbf{S}(\tilde{\Theta}), \quad h \in H(\tilde{\Theta}),$$

$$\rho(\tilde{S}) = \rho(T), \quad v(\tilde{S}) = v(S), \tag{7.17}$$

$$\text{INF}(h/S) = I(\eta, n_Z, P) \tag{7.18}$$

*Proof.* The relations  $\tilde{S} \in \mathbf{S}(\tilde{\Theta})$  and  $h \in H(\tilde{\Theta})$  are clear, and the equality

$$\rho(\tilde{S}) = \rho(T)$$

is proved straightforwardly:

$$\begin{aligned} \rho(\tilde{S}) &= \inf_{u \in U} \sum_{\tilde{\theta} \in \tilde{\Theta}} \tilde{L}(\tilde{\theta}, u) q(\tilde{\theta}) = \inf_{u \in U} \sum_{\substack{\theta \in \Theta \\ w \in W}} L(\theta, u) P(\theta) \mu(w/\theta) \\ &= \inf_{u \in U} \sum_{\theta \in \Theta} L(\theta, u) P(\theta) = \rho(T). \end{aligned}$$

In order to prove (7.17) we shall make use of Theorem refThm5.3:

$$v(\tilde{S}) = \rho(\tilde{S}) - \sum_{\tilde{\theta} \in \tilde{\Theta}} \left[ \inf_{u \in U} \tilde{L}(\tilde{\theta}, u) \right] q(\tilde{\theta})$$

$$\begin{aligned}
&= \rho(T) - \sum_{\substack{\theta \in \Theta \\ w \in W}} \left[ \inf_{u \in U} L(\theta, u) \right] P(\theta) \mu(w/\theta) \\
&= \rho(T) - \sum_{\theta \in \Theta} \left[ \inf_{u \in U} L(\theta, u) \right] P(\theta) = v(S).
\end{aligned}$$

Finally, to prove (7.18), note that since

$$\mathcal{D} = (W \rightarrow U),$$

the relationships

$$\begin{aligned}
\rho(\tilde{S}^h) &= \inf_{d \in D} \sum_{\tilde{\theta} \in \tilde{\Theta}} \tilde{L}[\tilde{\theta}, d(h(\tilde{\theta}))] q(\tilde{\theta}) = \inf_{d \in D} \sum_{\substack{\theta \in \Theta \\ w \in W}} L(\theta, d(w)) P(\theta) \mu(w/\theta) \\
&= \inf_{d \in D} \sum_{w \in W} \sum_{\theta \in \Theta} L(\theta, d(w)) P(\theta) \mu(w/\theta) \\
&= \sum_{w \in W} \inf_{u \in U} \sum_{\theta \in \Theta} L(\theta, u) P(\theta) \mu(w/\theta) \\
&= \sum_{w \in W} p(w; P, \eta) \inf_{u \in U} \sum_{\theta \in \Theta} L(\theta, u) P_\eta(\theta/w)
\end{aligned}$$

are valid, whence the required equality follows immediately. The theorem is proved.

## 7.4 A Generalization to Countable $\Theta$

It is natural to suppose that Theorems 7.1 and 7.2 are also valid in the case of a countable set  $\Theta$ . The proof of this case is given in [92] and is shorter than the previous one for finite  $\Theta$ , so we cite it here.

Let  $A$  be an arbitrary convex set. Let  $f_1$  and  $f_2$  be two real-valued functions defined on  $A$  and such that for arbitrary

$$\begin{aligned}
x_1, x_2, \dots, x_m, \dots \in A, \quad x'_1, x'_2, \dots, x'_n, \dots \in A, \quad a_1, a_2, \dots, a_m, \dots \in [0, 1], \\
a'_1, a'_2, \dots, a'_n, \dots \in [0, 1],
\end{aligned}$$

the relations

$$\sum_{i=1}^{\infty} a_i = \sum_{j=1}^{\infty} a'_j = 1, \quad \sum_{i=1}^{\infty} a_i x_i = \sum_{j=1}^{\infty} a'_j x'_j$$

imply that

$$\operatorname{sgn} \left[ \sum_i \alpha_i f_1(x_i) - \sum_j \alpha'_j f_1(x'_j) \right] = \operatorname{sgn} \left[ \sum_i \alpha_i f_2(x_i) - \sum_j \alpha'_j f_2(x'_j) \right].$$

In what follows, this statement will be written in the form  $(f_1, f_2) \in V(A)$ .

**Lemma 7.9.** *Let  $e_1, e_2, \dots, e_n, \dots$  be a collection of linearly independent vectors belonging to some convex set  $A$ . Let  $f$  and  $g$  be convex upward (concave) functions from  $A$  into  $\mathbb{R}$ , where  $(f, g) \in V(A)$  and  $f(e_i) = g(e_i) = 0$  for all  $i \in \mathbb{N}$ . Then  $\exists \lambda > 0$  such that*

$$f(x) = \lambda g(x) \quad \forall x \in S = [e - 1, e_2, \dots, e_n, \dots],$$

where

$$S = [e_1, e_2, \dots, e_n, \dots]$$

is the convex envelope of the vectors  $e_1, e_2, \dots, e_n, \dots$ .

*Proof.* Note that  $\forall x \in S, f(x) \geq 0$ . In fact,  $\forall x \in S \exists \{\alpha_i\}$  such that  $x = \sum_i \alpha_i \times e_i$ , and since the function  $f$  is convex upward, we have  $f(x) \geq \sum_i \alpha_i \times f(e_i)$  and hence  $f(x) \geq 0$ .

Let  $x \in S$  be such that  $f(x) > 0$ . We choose an arbitrary  $y \in S, y \neq x$  (if any), such that  $f(y) > 0$ .

We set

$$\gamma_1 = \frac{f(y)}{f(x) + f(y)} \quad \text{and} \quad \gamma_2 = \frac{f(x)}{f(x) + f(y)}.$$

It is obvious that for  $\gamma_1$  and  $\gamma_2$ , we have  $0 < \gamma_i < 1$  and  $\gamma_1 + \gamma_2 = 1$ . Let us consider the equation

$$\bar{x} = \gamma_1 x + \gamma_2 y.$$

We represent  $x$  and  $y$  in the form of their decomposition in terms of a system of vectors  $\{e_1, e_2, \dots, e_n, \dots\}$  that satisfies the following condition: there exist uniquely defined  $\{\alpha\}$  and  $\{\beta\}$  such that

$$\begin{aligned} x &= \alpha_1 e_1 + \dots + \alpha_n e_n + \dots, & \sum \alpha_i &= 1, \\ y &= \beta_1 e_1 + \dots + \beta_n e_n + \dots, & \sum \beta_i &= 1. \end{aligned}$$

Hence we can write

$$\bar{x} = \gamma_1 x + \gamma_2 (\beta_1 e_1 + \dots + \beta_n e_n + \dots) = \gamma_1 x + \gamma_2 \beta_1 e_1 + \dots + \gamma_2 \beta_n e_n + \dots,$$

and similarly,

$$\bar{x} = \gamma_1 \alpha_1 e_1 + \dots + \gamma_1 \alpha_n e_n + \dots + \gamma_2 y.$$

We have the equality

$$\gamma_1 + \gamma - 2\beta_1 = \cdots + \gamma - 2\beta_n + \cdots = \gamma_1 \alpha_1 + \cdots + \gamma_1 \alpha_n + \cdots + \gamma_2 = 1.$$

Since  $f(e_i) = 0 \quad \forall i \in N$ , we have

$$\begin{aligned} \gamma_1 f(x) + \gamma_2 \beta_1 f(e-1) + \cdots + \gamma_2 \beta_n f(e_n) + \cdots - \gamma_2 f(y) - \gamma_1 \alpha_1 f(e_1) - \cdots \\ - \gamma_1 \beta_n f(e_n) - \cdots = \gamma_1 f(x) - \gamma_2 f(y) = 0. \end{aligned}$$

Then  $(f, g) \in V(A)$  implies  $\gamma_1 g(x) - \gamma_2 g(y) = 0$ . Hence we have  $\gamma_1(x) = \gamma_2(y)$ . From this we obtain

$$\frac{g(x)}{f(x)} = \frac{g(y)}{f(y)}.$$

Denoting the ratio  $\frac{g(x)}{f(x)}$  by  $\frac{1}{\lambda}$ , we have  $f(t) = \lambda g(t)$  when  $t \in [x, y]$ . But since  $y$  is arbitrarily chosen from  $S$  and is such that  $f(y) \neq 0$ , for all  $x \in S$  such that  $f(x) \neq 0$  we have  $f(x) = \lambda g(x)$ .

Note that  $f(x) = 0$  implies  $g(x) = 0$ . In fact, let

$$x = \alpha_1 e_1 + \cdots + \alpha_n e_n + \cdots,$$

and then the condition

$$\begin{aligned} \text{sgn}[\alpha_1 f(e_1) + \cdots + \alpha_n f(e_n) + \cdots - f(x)] \\ = \text{sgn}[\alpha_1 g(e_1) + \cdots + \alpha_n g(e_n) + \cdots - g(x)] \end{aligned}$$

implies

$$\text{sgn}[f(x)] = \text{sgn}[g(x)],$$

and hence we have

$$f(x) = g(x) = 0.$$

From this we obtain

$$f(x) = \lambda g(x)$$

for an arbitrary  $x$  from  $S$ . The lemma is proved.

We turn now to the proof of Theorem 7.1 for a countable set  $\Theta$ .

*Proof (of Theorem 7.1).* A unique vector  $\bar{x} = (x_1, x_2, \dots, x_n, \dots)$  from  $S$ , where  $S$  is a simplex in the space  $l_1$ , corresponds to each distribution  $P$  from  $P(\Theta)$ . To each distribution concentrated at one point there corresponds some vector  $e_i$  in  $l_1$ , where

$$\begin{aligned} e_1 &= (1, 0, \dots, 0, \dots), \\ e_2 &= (0, 1, \dots, 0, \dots), \\ &\dots \\ e_n &= (0, 0, \dots, 1, \dots). \end{aligned}$$

Since the correspondence between  $P(\Theta)$  and  $S$  is one-to-one, we assume that the natural uncertainty function  $\gamma$  is defined on  $S$ , i.e., we have  $\gamma: S \rightarrow R$ . Let us consider the expression

$$\begin{aligned} & \text{sgn}[I(\eta_1, \nu, P) - I(\eta_2, \nu, P)] \\ &= \text{sgn}[I(\eta_1, \nu_2, \bar{x}) - I(\eta_2, \nu_2, \bar{x})] \\ &= \text{sgn}[\nu(\bar{x}) - \sum_{w' \in W_1} p(w'; \bar{x}, \eta_1) \nu(y'_w) - \nu(\bar{x}) + \sum_{w'' \in W_2} p(w''; \bar{x}, \eta_2) \nu(y''_w)] \\ &= \text{sgn}[\sum \beta_j \nu(y''_j) - \sum \alpha_i \nu(y'_i)], \end{aligned}$$

where

$$\beta_j = p(w''_j; \bar{x}, \eta_2), \quad \alpha_i = p(w'_i; \bar{x}, \eta_1), \quad \sum \beta_i = \sum \alpha_i = 1, \quad y''_i, y'_j \in S, \\ \sum \beta_j y''_j = \sum \alpha_i y'_i = \bar{x}.$$

Note that for any

$$\bar{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_n, \dots) \in S,$$

any sequence  $\{y_i\}_{i=1}^\infty, y_i \in S$ , and any  $i$ , there exist a distribution  $P \in P(\Theta)$  and a stochastic experiment  $(W, \mu)$  such that

$$p(w_i; P, \eta) = \lambda_i, \quad P_\eta(\theta_i/w_j) = y_{ij},$$

where  $y_i = (y_{i1}, y_{i2}, \dots, y_{in}, \dots)$ .

In fact, let us set

$$P(\theta_i) = \sum_j y_{ji} \lambda_j; \mu(w_i/\theta_j) = \frac{P_\eta(\theta_j/w_i) p(w_i; P, \eta)}{P(\theta_j)} = \frac{y_{ji} \lambda_i}{\sum_k y_{jk} \lambda_k}$$

for all  $j$  for which we have  $\sum_k y_{jk} \lambda_k > 0$ .

If the relation  $\sum_k y_{jk} \lambda_k = 0$  is true for some  $j$ , then this means that  $P(\theta_j) = 0$ , and hence it is natural to assume that

$$\forall i \mu(w_i/\theta_j) = 0.$$

It follows from the preceding reasoning that  $(\nu, \gamma_z) \in V(S)$ . It is easy to see that the relation  $(a, b) \in V(S)$  is an equivalence relation, and since

$$\left( \gamma_z, \inf_{u \in U} \sum_i L(\theta_j, u) x_i - \sum_i \left[ \inf_{u \in U} L(\theta_i, u) x_i \right] \right) \in V(S),$$

we have

$$\left( \nu, \inf_{u \in U} \sum_i L(\theta_i, u) x_i - \sum_i \left[ \inf_{u \in U} L(\theta_i, u) x_i \right] \right) \in V(S).$$

Using the lemma, we obtain

$$v(\bar{x}) = \lambda \left( \inf_{u \in U} \sum_i L(\theta_i, u)x_i - \sum_i \left[ \inf_{u \in U} L(\theta_j, u)x_i \right] \right),$$

where  $\lambda > 0$ . The theorem is proved.

We set

$$v_z(\bar{x}) = \inf_{u \in U} \sum_i L(\theta_i, u)x_i - \sum_i \left[ \inf_{u \in U} L(\theta_i, u)x_i \right], \quad \bar{x} \in S.$$

**Corollary 7.5.** *Let  $(\Theta, U, L)$  be a decision scheme, and let the set  $\Theta$  be countable. Then an arbitrary Bayesian uncertainty function  $v$  is nonnegative, concave (convex upward), and continuous in the natural topology generated by the norm  $\|\bar{x}\| = \sum_{i=1}^{\infty} |x_i|$  in the space  $l_1$  (recall that we identify the set  $P(\Theta)$  with the simplex  $S$  of the space  $l_1$ ).*

*Proof.* If  $(f_1, f_2)$  in  $V(A)$ , then the concavity of the function  $f_2$  on  $A$  implies the concavity of the function  $f_1$ . In fact, since  $f_2$  is concave, the inequality

$$\alpha f_2(x) + (1 - \alpha)f_2(y) \leq f_2(\alpha x + (1 - \alpha)y)$$

is true for all  $x, y \in A$  and for all  $\alpha \in [0, 1]$ , and since

$$\begin{aligned} & \text{sgn}[\alpha f_1(x) + (1 - \alpha)f_1(y) - f_1(\alpha x + (1 - \alpha)y)] \\ &= \text{sgn}[\alpha f_2(x) + (1 - \alpha)f_2(y) - f_2(\alpha x + (1 - \alpha)y)], \end{aligned}$$

we obtain the concavity of the function  $f_1$ . Next, since  $(v, \gamma_z) \in V(P(\Theta))$ , the concavity of  $\gamma_z$  implies the concavity of  $v$ . The nonnegativity of  $v$  follows from Theorem 7.1.

We now prove continuity. Since

$$v(\bar{x}) = \lambda (\gamma_z(\bar{x}) - \sum_i \inf_{u \in U} L(\theta, u)x_i),$$

and a linear function is always continuous, it suffices to prove the continuity of  $\gamma_z(\cdot)$ .

Let us consider some point  $\bar{x} \in S$  and some  $\delta$ -neighborhood

$$V_\delta = \{\bar{y} : \|\bar{x} - \bar{y}\| < \delta\}.$$

Then we have

$$|\gamma_z(\bar{y}) - \gamma_z(\bar{x})| = \left| \inf_u \sum_i L(\theta_i, u)y_i - \inf_u \sum_i L(\theta_i, u)x_i \right| \leq \left| \sum_i M y_i - \sum_i m x_i \right|,$$

where

$$M = \sup_{\theta, u} L(\theta, u), \quad m = \inf_{\theta, u} L(\theta, u), \quad \left| \sum_i M y_i - \sum_i m x_i \right| = \left| c^* \sum_i (y_i - x_i) \right|,$$

and where  $c^* = M - m$  and

$$\left| \sum_i (y_i - x_i) \right| \leq \sum_i |y_i - x_i| < \delta.$$

Hence we have

$$|\gamma_z(\bar{y}) - \gamma_z(\bar{x})| < c_* \delta.$$

This implies the pointwise continuity of  $\gamma_z(\cdot)$  and hence the continuity of  $v$ . The corollary is proved.

**Corollary 7.6.** *The function  $v : P(\Theta) \rightarrow R$  satisfies Condition 1 in Definition 7.2 of the natural uncertainty function if and only if there exists a positive  $\lambda$  such that the function  $v \rightarrow \lambda v_z$  is affine on  $P(\Theta)$ .*

*Proof.* Necessity. One can see from the proof of Theorem 7.1 that Condition 1 in Definition 7.2 of the natural uncertainty function is equivalent to  $(v, \gamma_z) \in V(P(\Theta))$ .

Let us introduce some notation. If  $x_1, x_2, \dots, x_n, \dots \in E$  are linearly independent vectors of a linear space  $E$ , and  $f$  is an arbitrary real-valued function defined on all of  $E$  or on a subset of  $E$  containing the points stated, then we can find an affine function  $g$  that is defined on the linear envelope of vectors  $x_1, x_2, \dots, x_n, \dots$  and assumes the value  $f(x_i)$  at points  $x_i$ .

We can specify this function by the formula

$$g(x) = \sum_{i=1}^{\infty} \alpha_i f(x_i),$$

provided that

$$x = \sum_{i=1}^{\infty} \alpha_i x_i, \quad \alpha_i \in \mathbb{R}.$$

We can now consider a function denoted by  $[f_{x_1, x_2, \dots, x_n, \dots}]$  and defined as follows. If  $A$  is the intersection of the domain of the function  $f$  and the linear envelope of the family of vectors  $x_1, x_2, \dots, x_n, \dots$ , then we can write

$$[f_{x_1, x_2, \dots, x_n, \dots}] : A \rightarrow R, \quad [f_{x_1, x_2, \dots, x_n, \dots}](x) = f(x) - g(x).$$

Let

$$[v_{P(\Theta)}] = [v_{P_{\theta_1}, P_{\theta_2}, \dots, P_{\theta_k}, \dots}].$$

It follows from Lemma 7.9 that

$$[v_{P_{\Theta}}] = \lambda v_z, \quad \lambda > 0.$$

Hence we obtain

$$v - \lambda v_z = v - [v_{P_\theta}],$$

i.e., according to the definition of the function  $[v_{P_\theta}]$ , the function  $v - \lambda v_z$  is affine on  $P(\Theta)$ .

Sufficiency. Let  $v - \lambda v_z$  be an affine function on  $P(\Theta)$ . Then for any

$$P_1, P_2, \dots, P_n, \dots \in P(\Theta), \quad P'_1, P'_2, \dots, P'_n, \dots \in P(\Theta), \quad \alpha_1, \alpha_2, \dots, \alpha_n, \dots [0, 1], \\ \alpha'_1, \alpha'_2, \dots, \alpha'_n, \dots [0, 1],$$

under the conditions

$$\sum_i \alpha_i = \sum_j \alpha'_j = 1, \quad \sum_i \alpha_i P_i = \sum_j \alpha'_j P_j,$$

we have

$$(v - \lambda v_z) \left( \sum_i \alpha_i P_i \right) = \sum_i \alpha_i (v - \lambda v_z) P_i = \sum_i \alpha_i v(P_i) - \sum_i \alpha_i \lambda v_z(P_i).$$

Then we have

$$(v - \lambda v_z) \left( \sum_j \alpha'_j P'_j \right) = \sum_j \alpha'_j v(P'_j) - \sum_j \alpha'_j \lambda v_z(P'_j),$$

whence we obtain

$$\sum_i \alpha_i v(P_i) - \sum_i \alpha_i \lambda v_z(P_i) = \sum_j \alpha'_j v(P'_j) - \sum_j \alpha'_j \lambda v_z(P'_j).$$

Thus we obtain that  $(v, \lambda v_z) \in V(P(\Theta))$ . At the same time, since  $(\gamma_z, \lambda v_z) \in V(P(\Theta))$ , taking into account the transitivity of the relation  $V$ , we have  $(v, \gamma_z) \in V(P(\Theta))$ , which means that Condition 1 is satisfied. The corollary is proved.

One can see from what has been said above that for each decision scheme  $Z = (\theta, U, L)$ , a unique (up to a positive multiplier  $\lambda$ ) uncertainty function  $v$  concave on  $P(\Theta)$  and continuous in the space  $l_1$  is defined. It turns out that the converse statement is also valid.

**Theorem 7.4.** *Let  $\Theta$  be a countable set, let the function  $v : P(\Theta) \rightarrow R$  be concave and continuous if it is considered as a function of  $l_1$ , and let this function be such that  $v(P) = 0$  for all  $P \in P(\theta)$ . Then there exists a decision system such that  $v$  is an uncertainty function for the system.*

*Proof.* We specify the mapping  $T$  from  $P(\Theta)$  into  $l_1$ ,

$$T : P \rightarrow P(\theta_1), P * \theta_2, \dots, P(\theta), \dots$$

It is obvious that  $T$  maps  $P(\Theta)$  in the simplex  $S$  generated by the vectors  $e_1, e_2, \dots, e_n, \dots$ . Let us introduce a function  $f$  from  $l_1$  into  $\mathbb{R}$ , which is specified by the formula

$$f(\bar{x}) = \begin{cases} -v(T^{-1}(\bar{x})), & \text{if } \bar{x} \in S, \\ +\infty, & \text{if } \bar{x} \notin S. \end{cases}$$

In this case, the function  $f(\bar{x})$  is a convex eigenfunction from  $\text{dom} F = S$ . Let us define the space of possible solutions  $U$  as the family of all affine functions  $u: l+1 \rightarrow R$  satisfying the condition  $u(\bar{x}) \leq f(\bar{x}) \forall \bar{x} \in l_1$ , namely,

$$U = \{u(\bar{x}) | u(\bar{x}) = \langle \bar{x}^*, \bar{x} \rangle + \alpha, \bar{x} \in l_1, \alpha \in R, u(\bar{x}) \leq f(\bar{x}) \in l_1\}.$$

We now specify a loss function  $L$  as follows:

$$l(\theta, u) = -u(T(P_0)),$$

where  $P_0$  is a distribution concentrated at the point  $\theta$ . Let us show that in this case, we have

$$\inf_{u \in U} \sum_i L(\theta_i, u) P(\theta_i) - \sum_i [\inf_{u \in U} l(\theta_i, u)] P(\theta_i) = v(P) \quad \forall P \in P(\Theta).$$

For each  $\bar{x} \in S$ , there exists a unique sequence  $\{\lambda_i(\bar{x})\}$  such that

$$\sum_i \lambda_i(\bar{x}) = 1, \quad 0 \leq \lambda_i(\bar{x}) \leq 1, \quad \sum_i \lambda_i(\bar{x}) e_i = \bar{x}.$$

Taking into account that the functions are affine, for all  $u \in U$  we have

$$u(\bar{x}) = \sum_i \lambda_i(\bar{x}) u(e_i),$$

and

$$\begin{aligned} \inf_{u \in U} \sum_i L(\theta_i, u) P(\theta_i) &= \inf_{u \in U} \sum_i -u(e_i) P(\theta_i) = \inf_{u \in U} \sum_i -\lambda_i(\bar{x}) u(e_i) \\ &= \inf_{u \in U} -u(\bar{x}) = -\sup_{u \in U} u(\bar{x}) = -f(\bar{x}) = v(P). \end{aligned}$$

The equality

$$-\sup_{u \in U} u(\bar{x}) = -f(\bar{x})$$

follows from the fact that a closed convex function can be specified as the supremum of affine functions. Let us choose an arbitrary  $\theta_i \in \Theta$  and consider

$$\inf_{u \in U} L(\theta_i, u) : \inf_{u \in U} L(\theta_i, u) = \inf_{u \in U} (-u(e_i)) = -\sup_{u \in U} u(e_i) = v(e_i) = 0.$$

Hence, we have

$$\inf_{u \in U} \sum_i L(\theta, u) P(\theta_i) - \sum_i [\inf_{u \in U} L(\theta_i, u)] P(\theta_i) = v(P).$$

Thus, we have a decision scheme for which the function  $v(\cdot)$  plays the role of an uncertainty function. The theorem is proved.

# Chapter 8

## Reducibility of Experiments in Multistep Decision Problems

Everything worth saying has already been said. But since no one was listening it is necessary to repeat it.

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André Gide

### 8.1 Preliminaries

The notion of informativity of experiment (Chapters 6, 7) is useful for the extension of some stochastic decision problems to random-in-a-broad-sense decision problems. One such extension that concerns a multistep decision problem is considered here. A multistep decision problem arises when the decision-maker has to make sequential decisions in the same situation [14]. If an experiment is included in this system, then sequential accumulation of information—decrease of uncertainty—becomes feasible [18]. Indeed, let  $n$  be the number of stages in the multistep problem. Let the decision-maker before making the  $(k + 1)$ th decision,  $k = 0, 1, \dots, n$ , perform an experiment (observation).

Since any experiment decreases (more precisely, does not increase) the uncertainty of the situation,<sup>1</sup> the sequence of such experiments may be used to construct the process of decrease of uncertainty. Such a sequential process is sometimes called an adaptive process. This process describes the adaptation of the decision-maker to the situation [7].

It turns out that this process depends essentially on the type of experiment, i.e., on what the decision-maker can observe in the experiment. Indeed, suppose that the decision-maker is dealing with a parametric situation, is using the matrix model, and before the  $(k + 1)$ th decision can measure directly the value of the unknown parameter  $\theta$ . Following the notation introduced in Chapter 7, we describe such an experiment by means of a random function  $\eta = (\eta_\theta)_{\theta \in \Theta}$  that

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<sup>1</sup> We consider only those experiments that possess such a feature.

takes values in some set  $W$ . In this experiment one can observe the value

$$w_k = \eta(\theta, \xi_k),$$

where  $\xi_k$  is the random error of observation. In Chapter 2 we called such an experiment an *experiment of the second type*, and in analogy to control theory, we shall call a decision system with such an experiment an *open-loop system*.

If the parameter  $\theta$  in a parametric situation cannot be measured directly or the decision-maker is dealing with a nonparametric situation, an experiment of the second type is impossible. In this case one can only measure the consequence  $c_k$  of one's action  $u_k$ . We shall describe such experiments by means of a random function  $f = (f_{c_k})_{c_k \in C}$  that takes values in some set  $Y$ . In this experiment—we called it an experiment of the first type—one can observe the value

$$y_k = f(c_k, \xi_k), \quad (8.1)$$

where  $\xi_k$  is the random error and

$$c_k = g(\theta, u_k). \quad (8.2)$$

It is convenient to call a decision system with an experiment of the first type, following again the terminology of control theory, a *closed-loop system*.

In both open- and closed-loop systems, observations  $w_k$  and  $y_k$  carry some information about the unknown that is used by the decision-maker to make the decision  $u_{k+1}$ , and further, to make decisions  $u_{k+2}, u_{k+3}, \dots, u_n$  [26]. But in the case of closed-loop systems, the observation  $y_k$  (and information contained therein) depends on the decision  $u_k$ . Namely

$$y_k = f(g(\theta, u_k), \xi_k).$$

So, in spite of the absence of dynamics (memory) in decision situations—in this book we consider only such situations—there is some influence of decision  $u_k$  on all future decisions in the decision system, a sort of *information dynamics*. There is no such effect of information dynamics in open-loop systems: observation  $w_k$  does not depend on decision  $u_k$ . This difference served as the basis for the following problem [18].

Define  $v_i = (y_i, u_i)$ ,  $i = 0, 1, \dots, k$ . Let  $v^k = (v_k, v_{k-1}, \dots, v_0)$  be the union of data that the decision-maker has before the  $(k-1)$ th decision, and  $V^k$  the set of all possible  $v^k$ . Further, let the decision  $u_{k+1}$  be chosen by the decision rule

$$\delta_k : V^k \longrightarrow U. \quad (8.3)$$

The collection  $(\delta_i, \dots, \delta_j) - \delta_i^j$  is called a strategy of the  $(i, j)$ -stage decision problem, and we have  $\delta_i \in \Delta_i^j$ , where  $\Delta_i^j$  is the set of all admissible strategies.

If the randomness in a decision system is stochastic, then the multistep decision problem consists in finding the best, or optimal, strategy  $\delta^* = \delta_k^*, k = C(1, n)$ , i.e., such that

$$E \left[ \sum_{k=1}^n L(\theta, \delta_k^*(v^k)) \right] = \min_{\delta \in \Delta_1^n} E \left[ \sum_{k=1}^n L(\theta, \delta_k(v^k)) \right]. \quad (8.4)$$

This optimization problem is known as the problem of stochastic dynamic programming [7]. In principle, this problem can be solved.

It was shown in [18] that there are decision systems with an experiment of the first type such that the corresponding multistep decision problem is reduced to a sequence of single-stage problems, i.e.,

$$\min_{\delta_1 \in \Delta_1^n} E \sum_{k=1}^n L(\theta, \delta_k(v^k)) = \sum_{k=1}^n \min_{\delta_k} E L(\theta, \delta_k(v^k)). \quad (8.5)$$

Such systems (and/or problems) were called *neutral*, and the property itself was called *neutrality* [18]. It is clear that an open-loop decision system is neutral because the observation  $w_k$  here does not depend on the decision  $u_k$ . However, it has been shown that there are closed-loop decision systems, i.e., decision systems with observation of consequences, that are neutral as well [18]. To study this class of systems is important. It is important not only because an early detection of neutrality of a decision system permits one to avoid unnecessary difficulties as well as the consequent coarsening of the problem (8.4) (see, for example, [33]). It is also essential that at the stage of modeling there is always some freedom in the choice of situation model, and if it is known that one of the competing models leads to a neutral system, then that is a strong argument in favor of such a model.

Unfortunately, the property of neutrality in closed-loop decision systems is stipulated by too many particulars, or reasons, and in the general case is not easy to recognize. However, these reasons can be divided into three classes, depending on the part of the problem to which they belong:

- (a) Neutrality stipulated by the structure of the situation and the structure of the experiment. This sort of neutrality holds for all probabilistic characteristics of the unknown parameter and experimental errors, and for all criteria of optimality.
- (b) Neutrality stipulated by a specific combination of the structure of the situation, the structure of the experiment, and their probabilistic characteristics. This neutrality holds for all criteria of optimality.
- (c) Neutrality results from the combination of all components of the decision system, and does not hold if criteria and probabilistic structure change.

The simplest to detect is neutrality of the class (a), which in [18] was called the *reducibility*. The following criterion for reducibility was formulated in [31]

and was proved in [34] for a wide class of Bayesian control systems. Decision systems described by equations (8.1) and (8.2) are reducible if and only if there exist functions  $\varphi(y_k, u_k)$  and  $\psi(\theta_k, \xi_k)$  such that

$$y_t = (g(u_k, \lambda_k), \xi_k) \Leftrightarrow \varphi(y_k, \xi_k) = \psi(\theta_k, \xi_k)$$

for all  $u_k, \theta_k, \xi_k, y_k$ .

In [34], it was proved that any reducible system is neutral, but the converse is not true. An example of a neutral but not reducible control system was proposed there, and one sufficient criterion for neutrality was obtained.

As a result, it became clear that a criterion for reducibility in the case of Bayesian decision systems defines the class of decision systems possessing the following characteristic property: regardless of the probabilistic properties of random values  $\theta_k$  and  $\psi_k$ , and regardless of the optimality criterion we choose, the decision system will be neutral (Theorem 8.1).

And in parallel, another fact became clear: the properties of reducibility and neutrality, as well as the notion of information accumulation rate, are essentially broader than one can observe when considering only Bayesian systems. This was the idea that generated the development of general notions of decision system and informativity of experiment. The purpose of this chapter is an application of the approach developed in previous chapters to the problem that generated this approach. Now this problem can be solved on a more general level (Theorem 8.2).

## 8.2 Reducibility of Bayesian Decision Problems

The study of neutrality and reducibility of a multistep Bayesian decision problem with experiment is based on our next theorem.

Let us first agree that if we speak about a measurable space  $X$ , we mean that  $X$  is a nonempty set and the  $\sigma$ -algebra  $A$  containing all one-point subsets of the set  $X$  is fixed. And if  $X$  and  $Y$  are two measurable spaces, then the symbols  $(X \rightarrow Y)$  and  $A_{X \times Y}$  denote respectively the families of all measurable functions from  $X$  to  $Y$  and the  $\sigma$ -algebra  $A_X \otimes A_Y$ .

**Theorem 8.1.** *Let  $U, Y, \Lambda, \Xi$  be measurable spaces, and we define the functions  $f \in ((\Lambda \times U \times \Xi) \rightarrow Y)$  and  $f(\Lambda \times \{u\} \times \Xi) \in A_Y$  for all  $u \in U$ . Then in order that the value*

$$\inf_{\delta \in (Y \rightarrow G)} E_p L(\xi, \delta(f(\xi, u, \chi)))$$

*not depend on  $u \in U$  for any measurable space  $G$ , any probability space  $(\Omega, A_\Omega, P)$ , and any  $\xi \in (\Omega \rightarrow \lambda)$ ,  $\chi \in (\Omega \rightarrow \Xi)$ ,  $L \in ((\lambda \times G) \rightarrow \mathbb{R})$ ,  $(\inf L(\Lambda \times G) > -\infty)$ , it is necessary and sufficient that there exist a set  $Q$  and functions*

$\varphi \in ((Y \times U) \rightarrow Q)$  and  $\psi \in ((\Lambda \times \Xi) \rightarrow Q)$  such that  $y = f(\lambda, u, \xi) \iff \varphi(y, u) = \psi(\lambda, \xi)$  for all  $\lambda \in \Lambda, u \in U, \xi \in \Xi, y \in Y$ .

The proof of this theorem relies on two lemmas.

**Lemma 8.1.** *If the set  $Q$  and the functions  $\varphi$  and  $\psi$  exist, then to every  $u \in U$  one can associate a measurable space  $Q_U$  and functions  $\varphi_U \in (Y \rightarrow Q_U)$  and  $\psi_U \in ((\Lambda \times \Xi) \rightarrow Q_U)$  such that the following conditions are satisfied:*

**Condition 1**  $\varphi_U$  is a single-valued mapping of  $Y$  in  $Q_U$ .

**Condition 2**  $Q_U \supseteq \psi(\Lambda \times \Xi)$ .

**Condition 3**  $\psi_U(\lambda, \xi) = \psi(\lambda, \xi)$  for all  $\lambda \in \Lambda, \xi \in \Xi$ .

**Condition 4**  $y = f(\lambda, u, \xi) \iff \varphi_U(y) = \psi_U(\lambda, \xi)$  for all  $\lambda \in \Lambda, u \in U, \xi \in \Xi, y \in Y$ .

*Proof.* Assume

$$\varphi_U : Y \rightarrow (Q \cup Y), \quad \varphi_U(y) = \begin{cases} \varphi(y, u), & \text{if } \varphi(y, u) \in \Psi(\lambda \times \Xi), \\ y, & \text{otherwise,} \end{cases}$$

$$Q_u = \varphi(Y).$$

We prove Condition 1. Let  $\varphi_u(y_1) = \varphi_u(y_2)$  and  $y_1 \neq y_2$ . This means that either

$$(1) \varphi_u(y_1) = \varphi_u(y_2) \in \psi(\Lambda \times \Xi)$$

or

$$(2) \varphi_u(y_1) = \varphi_u(y_2) \notin \psi_u(\Lambda \times \Xi).$$

But in case (1) we have

$$\Psi(\lambda_1, \xi_1) = \varphi(y_1, u), \quad \Psi(\lambda_2, \xi_2) = \varphi(y_1, u), \quad \varphi(y_1, u) = \varphi(y_2, u),$$

that is,

$$\varphi(y_1, u) = \Psi(\lambda_2, \xi_2),$$

and by virtue of the condition of Theorem 8.1,

$$y_1 = f(\lambda_2, u, \xi_2) = y_2.$$

In case (2),

$$\varphi_u(y_1) = y_1 \quad \text{and} \quad \varphi_u(y_2) = y_2$$

by the definition of the function  $\varphi_u$ , so that  $y_1 = y_2$ . Thus the mapping  $\varphi_u$  is single-valued, and with regard to the definition of  $Q_u$ , Condition 1 is proved.

Condition 2 follows from the fact that if  $q \in \Psi(\Lambda \times \Xi)$  and  $u \in U$ , then there exist  $\lambda \in \Lambda$ ,  $\xi \in \Xi$  such that  $q = \psi(\lambda, \xi)$ . Now taking  $y \in f(\lambda, u, \xi)$ , we have

$$\varphi(y, u) = \psi(\lambda, \xi) = q,$$

that is,  $\varphi_u(y) = q$ .

Condition 3 is obvious.

To prove Condition 4, note that  $\varphi(y, u) = \psi(\lambda, \xi)$  follows from  $y = f(\lambda, u, \xi)$ , i.e.,  $\varphi(y, u) \in \Psi(\Lambda \times \Xi)$ , and hence  $\varphi_u(y) = \psi(\lambda, \xi) = \psi_u(\lambda, \xi)$ .

Conversely, if  $\varphi_u(y) = \psi_u(\lambda, \xi)$ , then  $\varphi_u(y) \in \Psi(\Lambda \times \Xi)$ , and so  $\varphi_u(y) = \varphi(y, u) = \psi(\lambda, \xi)$ , i.e.,  $y = f(\lambda, u, \xi)$ .

Now in order for the functions  $\varphi_u$  and  $\psi_u$  to be measurable, it is sufficient to assume  $A_{Q_u} = \varphi_u(A_u)$ . Indeed, since the mapping  $\varphi_u$  is single-valued, it is automatically measurable. The mapping  $\psi_u$  is measurable on account of

$$\psi_u(\lambda, \xi) = \varphi_u(f(\lambda, u, \xi))$$

and the measurability of the functions  $\varphi_u$  and  $f$ . Lemma 8.1 is proved.

Now define

$$\Delta = (Y \rightarrow G), \quad W_u = (Q_u \rightarrow G), \quad u \in U.$$

**Lemma 8.2.** *If functions  $\varphi$  and  $\psi$  exist, then to any  $u \in U$  one can associate a single-valued mapping  $\alpha_u : \Delta \rightarrow W_u$ ,  $\alpha_u(\Delta) = W_u$ , such that*

$$E_p L(\xi, \delta(f(\xi, u, \chi))) = E_p L(\xi, W_u(\psi(\xi, \chi))) \quad (8.6)$$

for all  $\delta \in \Delta$ ,  $w_u = \alpha_u(\delta)$  and any probability state  $(\Omega, \mathfrak{A}_\Omega, P)$  and random elements  $\xi \in (\Omega \rightarrow \Lambda)$ ,  $\chi \in (\Omega \rightarrow \Xi)$ .

*Proof.* Set  $w_u(q) = \delta(\varphi_u^{-1}(q))$  for all  $\delta \in \Delta$ ,  $q \in Q_u$ . It is clear that the mapping  $\alpha : \delta \rightarrow W_u$  is single-valued. Indeed, from  $\delta_1 \neq \delta_2$  it follows that there exists  $y \in Y$  such that  $\delta_1(y) \neq \delta_2(\varphi_u^{-1}(q))$ , that is,  $\alpha_u(\delta_1) \neq \alpha_u(\delta_2)$ .

It is also clear that  $\alpha_u(\Delta) = W_u$ , since setting  $\delta'(y) = W'(\varphi_u(y))$  for any function  $w \in W_u$ , we have  $\delta' \in \Delta$ ,  $\alpha_u(\delta') = w$ . It remains to note that  $\delta(f(\lambda, u, \xi)) = \omega_u(\varphi_u(f(\lambda, u, \omega))) = W_u(\psi(\lambda, \xi))$  by definition of the functions  $\varphi_u$  and  $\psi$ , so equation (8.6) is correct. The lemma is proved.

*Proof (of Theorem 8.1).* Sufficiency. We first check that

$$\inf_{w \in W_u} E_p L(\xi, w(\psi(\xi, \chi)))$$

does not depend on  $u \in U$ . It is sufficient to check that for any  $u_1, u_2 \in U$ ,  $\omega_1 \in W_{u_1}$ , there exists  $w \in W_{u_2}$  such that  $w_2(q) = w_1(q)$  for all  $q \in \Psi(\Lambda \times \Xi)$ . Assume

that

$$w_2(q) = \begin{cases} w_1(q), & \text{if } q \in \psi(\Lambda \times \Xi), \\ q_0, & \text{if } q \in Q_{u_2} \setminus \psi(\Lambda \times \Xi), \end{cases}$$

where  $g_0$  is an arbitrary fixed element of the set  $G$ .

It is necessary to prove that the function  $w_2 : Q_{u_2} \rightarrow G$  is measurable. Let  $A \in A_G$ . If  $g_0 \in A$ , then

$$w_2^{-1}(A) = (Q_{u_2} \setminus \psi(\Lambda \times \Xi)) \cup w_1^{-1}(A).$$

If  $g_0 \notin A$ , then  $w_2^{-1}(A) = w_1^{-1}(A)$ , and we must show that

$$(Q_{u_2} \setminus \psi(\Lambda \times \Xi)) \in A_{Q_{u_2}}.$$

To this end it is enough to recall that

$$Q_u \setminus \psi(\Lambda \times \Xi) = Y \setminus f(\Lambda \times \{u\} \times \Xi).$$

The sufficiency is proved.

Necessity. Let  $Q, \varphi_u, \psi$  required by Theorem 8.1 not exist. Then one can find

$$(\lambda_0, \xi_0), (\lambda_1, \xi_1) \in \Lambda \times \Xi, \quad (y_1, u_1), (y_2, u_2) \in Y \times U$$

such that

$$y_1 = f(\lambda_0, u_1, \xi_0) = f(\lambda_1, u_1, \xi_1),$$

$$y_2 = f(\lambda_0, u_2, \xi_0) \neq f(\lambda_1, u_2, \xi_1) \stackrel{\text{def}}{=} y_3.$$

Assume

$$G = \{g_0, g_1\}, A_G = 2^G, \\ L(\lambda, g) = \begin{cases} 0, & \text{if } \lambda = \lambda_0, g = g_0 \text{ or } \lambda = \lambda_1, g = g_1, \\ 1, & \text{otherwise.} \end{cases}$$

Introduce now  $(\Omega, A_\Omega, P), \xi_u, \chi$  such that

$$P(\{\xi = \lambda_0, \chi = \xi_0\}) = \frac{1}{3}, \\ P(\{\xi = \lambda_1, \chi = \xi_1\}) = \frac{2}{3}.$$

Our goal is to produce a measurable function  $\delta^* \in (Y \rightarrow G)$  such that

$$E_p L(\xi, \delta^*(f(\xi, u_2, \chi))) < \inf_{\delta \in \Delta} E_p L(\xi, \delta(f(\xi, u_1, \chi))).$$

Assume

$$\delta^*(y) = \begin{cases} g_0, & \text{if } y = y_2; \\ g_1, & \text{otherwise.} \end{cases}$$

We obtain

$$\begin{aligned} E_p L(\xi, \delta^*(f(\xi, u_2, \chi))) &= \frac{1}{3}L(\lambda_0, \delta^*(y_2)) + \frac{2}{3}L(\lambda_1, \delta^*(y_3)) = 0, \\ E_p L(\xi, \delta(f(\xi, u_1, \chi))) &= \frac{1}{3}L(\lambda_0, \delta(t_1)) + \frac{2}{3}L(\lambda_1, \delta(y_1)) \\ &= \begin{cases} \frac{1}{3}, & \text{if } \delta(y_1) = q_1, \\ \frac{2}{3}, & \text{if } \delta(y_1) = g_0. \end{cases} \end{aligned}$$

Theorem 8.1 is proved.

### 8.3 Reducibility of a General Decision Problem

The purpose of this section is an application of the approach developed in Chapters 4 and 5 to the problem that generated this approach so that now this problem can be solved in a more general form.

Let

- (1)  $U, X, Y, \Lambda, \Xi$  be arbitrary sets;
- (2)  $g_1 : U \times \Lambda \rightarrow X, g_2 : X \times \Xi \rightarrow Y$  be arbitrary functions.

Interpreting  $g_1$  and  $g_2$  as described above, we consider the problem of reducibility in terms of decision problems.

In our considerations, the “vector”

$$((\lambda_{t+1}, \xi_{t+1}), (\lambda_{t+2}, \xi_{t+2}), \dots, (\lambda_N, \xi_N))$$

is unknown. Therefore, assuming without loss of generality  $t = 0$ , we set

$$\Theta = (\Lambda \times \Xi)^N.$$

In other words, we shall consider the utilization of information about  $\theta \in \Theta$  received at the moment  $t = 1$  for the control along the interval of time  $t = 2, \dots, N$ .

The decision that we have to choose is some “strategy” of control on this time interval, i.e., a “vector”  $(u_2, \delta_3, \delta_4, \dots, \delta_N)$ , where  $u_2 \in U, \delta_3 \in (Y \rightarrow U), \delta_4 \in (Y \times Y \times U) \rightarrow U$ , and so on, i.e.,  $\delta_n$  is a function that compares some sets of observations  $y_2, \dots, y_{n-1}$  and sets of control variables  $u_2, \dots, u_{n-1}$  with the control variable  $u_n$ . We denote the set of all such “vectors” by  $D$ . It is natural to

consider the properly recalculated control quality functional as the loss function  $L : \Theta \times D \rightarrow \mathbb{R}$ .

It remains to assign the information that we have about the behavior of the unknown parameter  $\theta$ . Assuming that this parameter is random in a broad sense, we formulate this information as some statistical regularity  $P \in \mathbf{P}(\Theta)$ ; see Chapters 4 and 5. As a result, we shall arrive at the decision problem

$$S = (\Theta, D, L, P).$$

The experiment that we are going to use for the choice of decision in this decision problem can be written as

$$h_u : \Theta \rightarrow Y, \quad h_u((\lambda_1, \xi_1), (\lambda_2, \xi_2), \dots, (\lambda_N, \xi_N)) = g_2(g_1(u, \lambda_1), \xi_2), \quad u \in U.$$

Neutrality of the corresponding decision system, i.e., independence of the information accumulation rate on the decision (strictly speaking, for the time being, only on the first step) will correspond to the equality

$$\text{INF}(h_{u_1}/S) = \text{INF}(h_{u_2}/S), \quad \forall u_1, u_2 \in U.$$

As was already noted in [34] and is seen from the present context, this property of a decision system can be induced by many causes, so that only the coarser property—in particular, the reducibility—can have practical significance. It is natural to introduce the following definition.

**Definition 8.1.** We shall say that a controlled process defined by equations

$$x_t = g_1(u_t, \lambda_t) \quad \text{and} \quad y_t = g_2(x_t, \xi_t) \quad (8.7)$$

is *reducible* if

$$\text{INF}(h_{u_1}/S) = \text{INF}(h_{u_2}/S)$$

for all  $u_1, u_2 \in U$  and  $S \in \mathbf{S}(\Theta)$ .

We have the following theorem.

**Theorem 8.2.** *The controlled process (8.7) is reducible if and only if there exist a set  $Q$  and functions*

$$\varphi : Y \times U \rightarrow Q \quad \text{and} \quad \psi : \Lambda \times \Xi \rightarrow Q$$

such that

$$y = g_2(g_1(u_1, \lambda), \xi) \Leftrightarrow \varphi(y, u) = \psi(\lambda, \xi) \quad (8.8)$$

for all  $y \in Y$ ,  $u \in U$ ,  $\lambda \in \Lambda$ ,  $\xi \in \Xi$ .

*Proof.* Necessity. It was noted in Chapter 6 that if the informativities of different experiments relative to all possible decision problems are equal, then all of these experiments generate the same classification. So, if the controlled process is reducible, then there exists a classification  $\mathcal{K}$  on  $\Theta$  that is generated by all  $u \in U$ . Suppose in this case  $Q = \mathcal{K}(\Theta) \cup \emptyset$  and define the function  $\varphi$  by the formula

$$\varphi(y, u) = h_u^{-1}(y) \quad (y \in Y, u \in U).$$

Obviously, if  $y = h_u(\theta)$ , then  $\mathcal{K}(\theta) = \varphi(y, u)$ . If, on the contrary, it is known that

$$\varphi(y, u) = \mathcal{K}(\theta),$$

then we have

$$y = h_u(\theta).$$

Thus, setting

$$\psi : \Lambda \times \Xi \rightarrow Q, \quad \psi(\lambda, \xi) = \mathcal{K}((\lambda_1, \xi_1), (\lambda_2, \xi_2), \dots, (\lambda_N, \xi_N)),$$

we obtain the required statement.<sup>2</sup>

Sufficiency. We have to prove that from the existence of  $Q$ ,  $\varphi$ , and  $\psi$  such that (8.8) holds, it follows that the equality

$$\mathcal{K}_{h_{u_1}} = \mathcal{K}_{h_{u_2}} \quad \forall u_1, u_2 \in U$$

holds as well. Obviously, this equality can be rewritten in the form

$$h_{u_1}(\theta^1) = h_{u_1}(\theta^2) \Rightarrow h_{u_2}(\theta^1) = h_{u_2}(\theta^2), \quad \forall u_1, u_2 \in U, \quad \theta^1, \theta^2 \in \Theta.$$

Therefore, given arbitrary

$$\theta^1, \theta^2 \in \Theta, \theta^j = ((\lambda_1^j, \xi_1^j), (\lambda_2^j, \xi_2^j), \dots, (\lambda_N^j, \xi_N^j)) \quad (j = 1, 2),$$

we set

$$h_{u_i} = h_i, y_i^j = g_2(g_1(u_i, \lambda^j), \xi^j) \quad (i, j = \{1, 2\}).$$

Then we obtain

$$h(\theta^j) = y_i^j,$$

and so from the equalities

<sup>2</sup> Here we have used the fact that  $h_u(\theta)$  for all  $u$ , along with  $K(\theta)$ , does not depend on  $(\lambda_2, \xi_2), (\lambda_3, \xi_3), \dots, (\lambda_N, \xi_N)$  as well.

$$h_1(\theta^1) = h_1(\theta^2), \quad \varphi(y_i^j, u_i) = \psi(\lambda^j), \xi_j \quad (i, j = \{1, 2\})$$

it follows that

$$y_1^1 = y_1^2 = y \in Y.$$

Thus we have

$$\varphi(y_1, u_1) = \psi(\lambda^1, \xi^1) = \psi(\lambda^2, \xi^2).$$

However, in such case,

$$\varphi(y_2^1, y_2) = \varphi(y_2^2, u_2) = \varphi(y, u_1) = \varphi(y_1^1, u_1)$$

immediately follows from

$$\varphi(y_2^j, u_2) = \varphi(\lambda^j, \xi^j) \quad (j = 1, 2).$$

In other words, simultaneously we have

$$\varphi(y_2^1, u_2) = \psi(\lambda^1, \xi^1) \quad \text{and} \quad \varphi(y_2^2, u_2) = \psi(\lambda^1, \xi^1).$$

But this means that  $y_2^1 = y_2^2$ , i.e.,  $h_2(\theta^1) = h_2(\theta^2)$ . The theorem is proved.



## Chapter 9

# Concluding Remarks

Decision-making under uncertainty accompanies all human activity, and so it is difficult to locate the beginnings of research in this field. At any rate, the consideration of problems in which nothing was known about the parameter but the set of its possible values—total uncertainty problems—can be found already in the work of Johann Bernoulli, Laplace, and Bayes alongside problems in which the unknown parameter was random with a given distribution. However, the “principle of insufficient foundation” (Bayes postulate) suggested therein, which in the case of total uncertainty recommended the consideration of all values of the unknown parameter as equiprobable, eventually proved to be logically inconsistent, and in 1854 it was subjected to serious criticism by J. Bull [29]. Attempts to save this principle by a formalization of conditions under which it does not lead to contradictions (see, for example, [10]) were undertaken later, but the overwhelming majority of researchers preferred to eschew the principle entirely, and as a result, quite a number of approaches to problems with uncertainty appeared.

Among contemporary directions, the one based on the “principle of guaranteed result” has the largest number of adherents. One of the bases of this principle is the idea that if in the structure of the “operating entity” one can distinguish between a “decision-making person” (DMP) and an “operations researcher” (OR), then it is quite natural to assume that only the DMP has the right to take risks. Hence, getting from the OR the decision problem (in our terminology  $\Theta$ ,  $U$ ,  $L$ , and “the information about  $\Theta$ ”), the DMP ought to focus on the worst “opponent’s response” under the given conditions, for otherwise he could well arbitrarily substitute the information about  $\Theta$  provided to him by the OR, and thereby expose the whole “operating entity” to the dangers of risk. In the simplest case, when nothing is known about  $\Theta$ , the principle of guaranteed result leads to the well-known minimax (maximin) criterion. This case is the one most studied at the present time (see, for example, [48, 52, 63]).

There is a more flexible construction in which two uncontrolled parameters of a different nature are considered: the first behaves in a random manner with a given distribution, and about the second we know only the set of its possible

values. The criterion of the average loss minimization, appearing naturally in this situation, has been studied in a large number of works (see, for example, [70, 79]). The question of the universality of such a presentation of information about the unknown parameter has been studied in [37, 38, 56].

Another approach, now in active development, is based on the desire to obtain the missing a priori distribution from some line of reasoning and then to use Bayesian methods. Two branches of this approach should be noted: on the one hand, the work of a subjectivist school in probability theory [73, 28, 30], and on the other hand, the work of a fiducial school in statistics; see [45] and references therein.

There exists one more approach, which in some sense is opposite to the first. Its supporters do not divide the “operating party” strictly into MDP and OR. In this approach, the analysis and ordering of the “information situations” actually encountered, as well as of the criteria used in them, is conducted in order to provide the operating party with the possibility of choice of criterion in every concrete situation. The authors of [80, 79] and many others adhere to this direction.

The simultaneous development of such different approaches to decision-making problems under uncertainty suggests that each of them has a certain range of application. The so-called axiomatic approach to optimality principles [3, 64, 65, 82] is dedicated to attempts to depict the contours of these ranges. This approach consists not in a formulation of some definite optimality principle but rather in the specification of the set of conditions to which the unknown principle should be subordinated. It turns out that in some cases, the quite restricted set of natural requirements of “reasonability” leads to a unique optimality principle satisfying these requirements, or to a rather narrow class of such principles. The theorems of expected and nonexpected utilities are now the best-known examples of this approach [59].

Different versions of a formal description of the problem that is often called the “game against nature” establish an important direction of the axiomatic approach. We note here that researchers began to study this kind of uncertainty earlier than all other kinds. The formulation of the “principle of insufficient foundation” already implied that the value of the unknown parameter “is chosen” by nature, but not by a conscious adversary; otherwise, it would be unlikely that its behavior could be described by a uniform distribution. In this sense, one can consider the problems of mathematical statistics as the “game against nature” [8]. As a matter of fact, before the emergence of game theory, mathematics did not consider problems of competing against a conscious opponent, though there were attempts at a mathematical description of social phenomena, and hence the absence of such an opponent did not need to be mentioned. At the present time, the successes achieved by game theory have led to an increased effort to apply its results to the “game against nature.” A large body of work (see, for example, [3, 12, 30, 58, 82]), essentially of the axiomatic direction, is dedicated to this

endeavor. These works do not answer the question about the behavior of nature, but rather are attempts to describe the set of conditions that a reasonable choice of decision under such conditions has to satisfy.

An alternative approach was proposed in [22]. There it was pointed out that “excessive reinsurance,” which usually is linked to the application of the principle of guaranteed result in the “games against nature,” does not follow from this principle itself, and appears only out of one’s inability to take into account all available information. From this point of view, the problem is to provide formal restrictions on the set of possible “movements” of nature, reflecting its “indifference.” The first attempts in this direction were made, as far as we know, in [36], and later this approach was developed in [54, 56].

The search for a description of “mass” but nonstochastic phenomena began as well not long ago (see [36, 37, 39, 54]). Although it was well known [47] that the concept of randomness used in everyday life is essentially wider than the concept of stochasticity used in probability theory, the efforts of researchers in this field were, beginning with [30], concentrated on finding conditions that would guarantee stochasticity of the considered phenomena (see [89] and references therein).

As distinct from what has been said about decision-making problems under uncertainty, interest in the problem of measuring the informativity of an experiment (in other terminology, the information quantity, the information precision, or the value or price of information contained in an experiment) appeared comparatively recently. As is well known, mathematical information theory originated in the classical work of C. Shannon, completed after Second World War. Considerable success achieved by this theory in the postwar decades nurtured the hope that a universal method of measurement of the information quantity contained in any report or any message had been discovered. The problems of fields such as mathematical statistics, control theory, physics, and biology were studied from the point of view of information theory (see [50, 49, 9, 28, 57] and also the references in [85]).

However, at the beginning of the 1960s it was noted that in the majority of applications, only two properties of Shannon’s entropy—its nonnegativity and concavity on the set of probability distributions—were used. Therefore, M. DeGroot proposed to call any function or distribution possessing these two properties an uncertainty function and to consider the average decrease of uncertainty in the result of experiment as the “information quantity contained in the experiment” [14]. This was the beginning of a new branch of information theory that was actively developed in the following years; see, for example, [77, 27, 1, 16, 57]. A large variety of uncertainty functions, natural in one or another situation, was studied, and there was considerable work on the classification of information situations and appropriate uncertainty functions [79].

Yet the question of the extent of arbitrariness in the choice of the uncertainty function for every concrete information situation remained open. At approxi-

mately the same time, some studies of the rate of accumulation of information in adaptive control systems were done in the field of automatic control theory [18, 31]. The specificity of the considered problem consisted in the fact that the question of information measurement was essentially secondary. In any event, the decision about the choice of control was dictated by the wish to decrease average losses irrespective of the information measurement method. If, for example, for some fixed information measurement method we had greater losses when using a more informative experiment, we would rather have refrained from the information-measurement method, but we would entertain no doubt about the wish to decrease average losses. From this point of view, the requirements of a “reasonability” of the information measure became more rigid, and this allowed one to obtain [35, 56] theorems of existence and uniqueness of the permissible uncertainty functions in Bayesian systems. Thus the informativity of stochastic experiment turned out to depend on the Bayesian system and to coincide to within the scale factor with the previously studied “value of information” [62, 77, 27]. A similar expression for the more general case of decision system was studied in [56, 40].

In conclusion, we remark that the title of this book contains two terms—“decision system” and “nonstochastic randomness”—that have been rarely if ever used in the previous literature. However, the first of these terms denotes, for the purpose of brevity, a well-known notion. Indeed, the axiomatics that defines the criterion choice rule, while being a certain model of the decision-maker, at the same time always contains, completely or partially, the model of the decision-making situation. But it is precisely this pair that constitutes what is called a decision system. The second term—nonstochastic randomness—is used in order to denote so-called statistically unstable random phenomena. This type of behavior of the cause–effect mechanism of a decision-making situation fills the interval between statistically stable behavior and behavior about which the decision-maker knows absolutely nothing. The question of existence of regularities of nonstochastic randomness was formulated in [47]. In [39] the theorem of existence of so-called statistical regularities of nonstochastic randomness was proven. This makes the construction of such regularities possible and useful in describing real situations of decision-making.

# Appendix A

## Mathematical Supplement

### A.1 Elements of the Theory of Binary Relations

#### A.1.1 Basic Operations on Sets

Let us recall the definitions of some operations that are often performed on sets, and define the sets resulting from these operations. The union (or sum) of two sets  $A$  and  $B$  is the set

$$A \cup B$$

consisting of all elements that belong to at least one of the sets  $A$  and  $B$ .

The union of any number of sets is defined analogously: if we are given an arbitrary indexed family of sets, that is, a collection of sets  $A_\alpha$  in which the index  $\alpha$  runs through an arbitrary set (the *index set*), the union of the sets  $A_\alpha$  is denoted by

$$\bigcup_{\alpha} A_\alpha.$$

By definition, this union consists of all elements that belong to at least one of the sets  $A_\alpha$ . In addition, we agree that if the domain of the index  $\alpha$  is not indicated, then the union extends over all the values that  $\alpha$  may take in the given problem. The same refers to the operation of intersection.

The intersection of two sets  $A$  and  $B$  is the set

$$A \cap B$$

that consists of all elements belonging to both  $A$  and  $B$ . Analogously, the intersection of an arbitrary (finite or infinite) number of sets  $A_\alpha$  is the set

$$\bigcap_{\alpha} A_\alpha$$

of all elements that belong to all of the sets  $A_\alpha$ .

The difference of the sets  $A$  and  $B$  is the set

$$A \setminus B$$

consisting of those elements in  $A$  that are not contained in  $B$ . It is not assumed here that  $B \subset A$ . If  $B \subset A$ , then the set  $A \setminus B$  is called the complement of the set  $B$ . It is clear that the complement to the set  $A \setminus B$  is the set  $B$  itself.

The following relations can be easily verified:

(1)

$$A \setminus B = A \setminus (A \cap B).$$

(2) If  $B \subset A$  (and only in this case)

$$(A \setminus B) \cup B = A.$$

(3) For any family of sets  $A_\alpha$  the equation

$$\left( \bigcup_{\alpha} A_{\alpha} \right) \cap B = \bigcup_{\alpha} (A_{\alpha} \cap B) \quad (\text{A.1})$$

(the distributive law) holds. In other words, the operation of intersection distributes over union.

(4)

$$\left( \bigcap_{\alpha} A_{\alpha} \right) = \bigcap_{\alpha} (A_{\alpha} \cup B). \quad (\text{A.2})$$

This is another form of the distributive law.

(5)

$$(A \setminus B) \cap C = (A \cap C) \setminus (B \cap C).$$

(6) If  $A = \bigcup_{\alpha} A_{\alpha}$  and  $B$  is an arbitrary set, then

$$A \setminus B = \bigcup_{\alpha} (A_{\alpha} \setminus B).$$

(7) If the sets  $B_\alpha$  are subsets of the set  $A$ , and  $C_\alpha$  is their complement, that is,  $C_\alpha = A \setminus B_\alpha$ , then

$$A \setminus \bigcup_{\alpha} B_{\alpha} = \bigcap_{\alpha} C_{\alpha} \quad \text{and} \quad A \setminus \bigcap_{\alpha} B_{\alpha} = \bigcup_{\alpha} C_{\alpha}. \quad (\text{A.3})$$

Two sets  $A$  and  $B$  are called disjoint (or nonintersecting) if their intersection is empty:  $A \cap B = \emptyset$ . If some system of sets  $A_\alpha$  is given and any two index sets belonging to this system and having different indices are disjoint, that is,

$$A_{\alpha_1} \cap A_{\alpha_2} = \emptyset \quad \text{for } \alpha_1 \neq \alpha_2,$$

then the sets  $A_\alpha$  are said to be pairwise disjoint (or simply disjoint).

It can be easily verified that the following auxiliary relations are correct:

(i) If the sets  $A_i$  ( $i = 1, 2, \dots$ ) form a decreasing sequence, that is,

$$A_1 \supset A_2 \supset \dots \supset A_i \supset \dots,$$

and if

$$\bigcap_{i=1}^{\infty} A_i = \emptyset,$$

then

$$A_1 = (A_1 \setminus A_2) \cup (A_2 \setminus A_3) \cup \dots = \bigcup_{i=1}^{\infty} (A_i \setminus A_{i+1}).$$

It is obvious that the sets  $A_i \setminus A_{i+1}$  are disjoint.

(ii) If  $A_i$  are disjoint sets and

$$B_n = \bigcup_{i=n+1}^{\infty} A_i,$$

then

$$\bigcap_{n=1}^{\infty} B_n = \emptyset.$$

Moreover, it is obvious that the sets  $B_n$  form a decreasing sequence.

(iii) If

$$A = \bigcup_{i=1}^{\infty} A_i,$$

then

$$\begin{aligned} A &= A_1 \cup [A_2 \setminus A_1] \cup [A_3 \setminus (A_1 \cup A_2)] \cup \dots \cup [A_i \setminus \bigcup_{j=1}^{i-1} A_j] \cup \dots \\ &= \bigcup_{i=1}^{\infty} [A_i \setminus \bigcup_{j=1}^{i-1} A_j] \end{aligned} \quad (\text{A.4})$$

(if  $i = 1$  it is natural to put  $\bigcup_{j=1}^{i-1} A_j = \emptyset$ ). It is clear that the sets  $A_i \setminus \bigcup_{j=1}^{i-1} A_j$  are disjoint.

The formula (A.4) becomes simpler if the sets  $A_i$  form an increasing sequence, that is,

$$A_1 \subset A_2 \subset \cdots \subset A_i \subset \cdots .$$

Then

$$\bigcup_{j=1}^{i-1} = A_{i-1}$$

and formula (A.4) turns into

$$A = A_1 \bigcup (A_2 \setminus A_1) \bigcup \cdots \bigcup (A_i \setminus A_{i-1}) \bigcup \cdots . \quad (\text{A.5})$$

### A.1.2 Topological Spaces

Let  $E$  be a nonempty set the elements of which are called points.

**Definition A.1.** The system  $S^*(x)$  of the subsets  $S(x)$  of the set  $E$  is called a *system of neighborhoods*, and its elements  $S(x)$  are called the neighborhoods of the point  $x$  if the following conditions are satisfied:

**Condition 1** Any neighborhood of the point  $x$  contains this point, that, is  $x \in S(x)$ .

**Condition 2** The intersection of any two neighborhoods of the point  $x$  is a neighborhood of the point  $x$ .

**Condition 3** Any set that contains a neighborhood of the point  $x$  is a neighborhood of the point  $x$ .

**Condition 4** Any neighborhood of the point  $x$  contains a set that is a neighborhood of each of its points as well as of the point  $x$ .

**Definition A.2.** A point  $x$  is called an *inner point* of  $X$  if  $X$  is a neighborhood of  $x$ . If every neighborhood of  $x$  contains at least one point from  $X$ , then  $x$  is called a *point of tangency* of  $X$ . The set of all inner points of  $X$  is called the *interior* and is denoted by  $\text{Int}(X)$ . The set of all tangency points of  $X$  is called the *closure* and is denoted by  $\text{Cl}(X)$ . The set  $\text{Fr}(X) = \text{Cl}(X) \setminus \text{Int}(X)$  is called the *boundary* (or *frontier*) of  $X$ .

**Definition A.3.** The set  $X = \text{Cl}(X)$  is said to be *closed*. A set that is a neighborhood of every point ( $X = \text{Int}(X)$ ) is said to be *open*. The system of all open subsets of the set  $E$  is called a *topology* of  $E$  and is denoted by  $T(E)$ . The set  $E$  along with the topology (an ordered pair  $(E, T(E))$ ) is called a *topological space*.

**Theorem A.1.** The union of any number of open sets is an open set. The intersection of a finite number of open sets is an open set. The space  $E$  and the empty set  $\emptyset$  are open.

**Theorem A.2.** *The set  $X$  is closed if and only if its complement is open. The empty set  $\emptyset$  and the set  $E$  are open and closed simultaneously.*

We have the following relations:

$$\begin{aligned} \text{Cl}(X \cup Y) &= \text{Cl}(X) \cup \text{Cl}(Y), \\ \emptyset &= \text{Int}(\emptyset) = \text{Cl}(\emptyset), \\ \text{Int}(X \cap Y) &= \text{Int}(X) \cap \text{Int}(Y), \\ E &= \text{Int}(E) = \text{Cl}(E), \\ \text{Int}(\text{Int}(X)) &= \text{Int}(X) \subset X \subset \text{Cl}(X) = \text{Cl}(\text{Cl}(X)), \\ \text{Fr}(X) &= \text{Cl}(X) \setminus \text{Int}(X) = \text{Cl}(X) \cap \text{Cl}(C(X)) \\ &= \text{Cl}(\text{Int}(X) \cup \text{Int}(\text{Cl}(X))). \end{aligned}$$

On a given set  $E$  one can define different systems of open sets, thus defining different topological spaces. A topology  $T_1(E)$  is said to be *stronger* than  $T_2(E)$  if  $T_1(E) \supset T_2(E)$ . The discrete topology  $T^*(E) = \exp E$  is the strongest, while the topology  $T_0(E) = (E, \emptyset)$  is the weakest.

Let  $E_1 \in E$ , and let  $T(E_1) = (E_1 \cap X_l : l \in L)$  be the intersection of  $E_1$  with all open sets from  $E$ . The ordered pair  $(E_1, T(E_1))$  can be considered as a topological space. If  $S_{T(E)}^*(x)$  is a system of neighborhoods of the point  $x$  in the topological space  $(E, T(E))$ , then the system of neighborhoods  $S_{T(E_1)}^*(x)$  of the same point in the space  $(E_1, T(E_1))$  is determined as the intersection of the element from  $S_{T(E)}^*(x)$  with the set  $E_1$ . In this case the topology  $T(E_1)$  is called the trace of the topology  $T(E)$  on  $E_1$ , and  $S_{T(E_1)}^*(x)$  is called the trace of the neighborhoods of  $x$  on  $E_1$ .

**Definition A.4.** A topological space  $(E_1, T(E_1))$ , where  $T(E_1)$  is the trace of the topology of the space  $(E, T(E))$  on  $E_1$ , is called a *subspace* of  $E$ .

If it is necessary to indicate in which topology the sets are considered; the notation  $\text{Cl}_T(X)$ ,  $\text{Int}_T(X)$ ,  $\partial \text{Fr}_T(X)$ , etc. is used.

**Theorem A.3.** *The topology  $T_1$  is stronger than the topology  $T_2$  if and only if every neighborhood of an arbitrary point  $x \in E$  in the topology  $T_2$  is a neighborhood of that point in the topology  $T_1$ .*

**Definition A.5.** A system  $G^*$  of open sets of the topological space  $(E, T)$  is called a *basis* of this space if every open set from  $T$  can be represented as a union of the sets from  $G^*$ .

**Definition A.6.** The set  $X$  is called *dense* in the set  $Y$  if  $\text{Cl}(X) \supseteq Y$ . If the set  $X$  is dense in  $E$ , then it is called *everywhere dense*. A closed set is called *nowhere dense* in  $E$  if its complement is everywhere dense. A topological space is called *separable* if it has a countable everywhere dense subset.

**Definition A.7.** If every pair of distinct points of the topological space  $(E, T)$  have neighborhoods that do not intersect, then the space is called a Hausdorff space. A Hausdorff space in which any two disjoint closed sets have disjoint neighborhoods is called *normal*.

**Definition A.8.** The topological product of two spaces  $(E_1, T_1)$  and  $(E_2, T_2)$  is the space  $(E, T)$  such that  $E = E_1 \times E_2$  and the topology  $T(E)$  consists of all possible unions of the direct products of the open spaces from  $E_1$  and  $E_2$ .

### A.1.3 Binary Relations

Let  $M$  be an arbitrary set. Consider the set of all pairs  $(x, y)$ , where  $x$  and  $y$  are both elements of  $M$ . We shall consider such pairs to be ordered, that is, we shall distinguish the pair  $(x, y)$  from the pair  $(y, x)$ . The set of such ordered pairs is called the *Cartesian product*  $M \times M$ .

A binary relation on  $M$  is a subset  $A$  of the set  $M \times M$ . If the pair  $(x, y)$  belongs to  $A$ , we write  $xAy$ , which can be read as “ $x$  is in the relation  $A$  with  $y$ .” We shall call expression  $xAy$  itself a *relation*.

More formally, a binary relation is an ordered pair  $(A, M)$ , where  $A \subseteq M \times M$ . We shall call the set  $M$  the *domain of definition* of the binary relation, and the set  $A$  of the pairs the *graph* of the relation.

We now define a binary relation on a finite set using matrices. If we enumerate the elements of the set  $M = \{x_i, i = 1, 2, 3, \dots, n\}$ , then the square matrix with the elements

$$a_{ij} = \begin{cases} 1, & \text{if } x_iAx_j \text{ is true,} \\ 0, & \text{if } x_iAx_j \text{ is false,} \end{cases} \quad (\text{A.6})$$

defines the graph of the relation  $A$ .

There is another important way to define binary relations on finite sets. We can represent the elements of the set  $M$  as the points in a plane. If the binary relation  $x_iAx_j$  is true, we join the elements  $x_i$  and  $x_j$  by an arrow. If  $x_iAx_i$  is true, then we plot a loop connecting the point  $x_i$  to itself. In other words, we define our binary relation as an oriented graph.

If the relation  $A$  on the set  $M$  is such that for any  $x \in M$  there is only one element  $y \in M$  for which  $xAy$  is true, then this relation is called a *function* or a *mapping* or a *single-valued mapping*. The element  $y \in M$  that corresponds to the element  $x \in M$  is called the *value* of the function  $A$  on the element  $x$ . This dependence between  $x$  and  $y$  is denoted by

$$y = A(x).$$

Often we need to consider a more general case of a binary relation between the elements of two different sets  $M$  and  $L$ . Such a relation is a subset of the set  $M \times L$ . Formally, this relation is defined as a triple  $(A, M, L)$ , where  $A \subseteq M \times L$ .

In this case we shall again call the relation  $A$  a function if for every  $x \in M$  there is only one element  $y \in L$  for which  $xAy$  is true. We shall denote such a function by  $A : M \rightarrow L$ ; here  $M$  is called the domain of definition of the function  $A$ , while  $L$  is its domain of values. The mapping  $A : M \rightarrow L$  is also called a mapping of the set  $M$  into the set  $L$ . The element of the set  $L$  that corresponds to the element  $x \in M$  is denoted by  $A(x)$  and is called the image of the element  $x$ . The element  $x$  itself is called the preimage of the element  $A(x)$ .

A mapping  $A : M \rightarrow L$  is called *surjective* if every element  $y \in L$  has a preimage. In this case one also says that  $M$  is mapped *onto*  $L$ .

The mapping  $A : M \rightarrow L$  is called *injective* if for every element  $y \in L$  there is no more than one preimage.

If the mapping  $A : M \rightarrow L$  is surjective and injective simultaneously, it is called *bijective*. If there is a bijective mapping  $A : M \rightarrow L$  for sets  $M$  and  $L$ , then they are called equivalent (of equal strength).

It is useful to introduce the following notions.

Let  $A : M \rightarrow L$  and let  $M_1$  be a subset of  $M$ . The set  $A(x)$  of images of all  $x \in M_1$  is called the image of the set  $M_1$  (it is denoted by  $A(M_1)$ ). In particular,  $A(M)$  is the image of the whole of  $M$ . It is easy to see that  $A : M \rightarrow A(M)$  is a surjective mapping.

Similarly, if  $L_1 \subseteq L$ , then the union of the preimages of all elements from  $L_1$  is called the complete preimage of the set  $L_1$  and is denoted by  $A^{-1}(L_1)$ .

Let us define now the *identity mapping* of the set  $M$ :

$$\mathcal{E}_M : M \rightarrow M.$$

Let  $A : M \rightarrow L$ . The mapping  $B : L \rightarrow M$  is called the *inverse* to  $A$  if  $AB = \mathcal{E}_M$  and  $BA = \mathcal{E}_L$  (where here the product of mappings denotes their composition or their consecutive application from right to left), that is, if the mapping  $A$  transfers any image  $A(x)$  in  $x$  and the mapping  $B$  transfers any image  $B(y)$  in  $y$ . We shall write in this case

$$B = A^{-1}.$$

It is easy to check that the inverse mapping exists if and only if the mapping itself is bijective.

Sometimes it is convenient to consider the functions  $A : M \rightarrow L$  that are defined not on the whole of  $M$  but only on a subset  $M_1$ , which then is called the domain of definition of the function. Then it is useful to complete the set  $L$  to the set  $L_{\#} = L \cup \{\#\}$ , adding to  $L$  the element  $\{\#\}$  previously not belonging to  $L$ . The element  $\{\#\}$  plays the role of an empty element. In this case we say (by definition) that the mapping  $A : M \rightarrow L_{\#}$  gives a correspondence between any element from  $M \setminus M_1$  and the empty element  $\{\#\}$ . Often it is convenient to suppose

that every set contains an empty element. Then we do not have to distinguish between  $L$  and  $L_{\#}$ .

For an arbitrary relation  $(A, M, L)$ , the elements  $y$  such that  $xAy$  are said to *correspond* or *be associated* with the element  $x$ . In these cases, we shall call the relation  $(A, M, L)$ , which acquires, so to speak, a functional character, a *correspondence*. Thus a correspondence is a many-valued function (many-valued mapping). The notation  $\Psi : M \rightarrow L$  for an arbitrary relation  $\Psi = (A, M, L)$  shall imply that we consider the relation  $\Psi$  to be a correspondence.

We could, however, consider, instead of the correspondence  $\Psi : M \rightarrow L$ , the function  $A : M \rightarrow 2^L$  that associates to any element  $x \in M$  the set  $L_x \subseteq L$  of all  $y$  for which  $(x, y) \in A$  (in particular,  $L_x$  can be the empty set). Nevertheless, the language of correspondences (of many-valued mappings) is often more practical.

As in the case of “single-valued” functions, one can introduce the notions of a correspondence that is everywhere defined (for any  $x \in M$  the set  $L_x$  is nonempty), of an injective correspondence ( $L_x \cap L_y = \emptyset$  for any  $x \neq y$ ), and of a surjective correspondence (for any  $y \in L$  there exists an  $x \in M$  for which  $y \in L_x$ ).

### ***A.1.4 Operations on Binary Relations***

Starting from the operations on sets, we can define operations on relations. In this section we shall assume that all relations are defined on the same set  $M$ .

Let us take two relations  $A$  and  $B$ . Each of them corresponds to a certain number of pairs (subsets  $A \subseteq M \times M$  and  $B \subseteq M \times M$ ).

The intersection of two relations  $A$  and  $B$  is the relation  $A \cap B$  defined by the intersection of the corresponding sets. It is clear that the relation  $x(A \cap B)y$  holds if and only if two relations  $xAy$  and  $xBy$  are true simultaneously.

The union of two relations  $A$  and  $B$  is the relation  $A \cup B$  defined by the union of the corresponding sets. It is clear that the relation  $x(A \cup B)y$  is true if and only if at least one of the two relations  $xAy$  and  $xBy$  is true.

The negation of the relation  $A$  is the relation  $\bar{A}$  defined by the complement of the set  $A$  in  $M \times M$ , that is,  $\bar{A} = (M \times M) \setminus A$ . The notation  $x\bar{A}y$  (it is not true that  $x$  and  $y$  are in the relation  $A$ ) means that  $(x, y) \notin A$ , which is equivalent to  $(x, y) \in (M \times M) \setminus A$ .

It is clear that for any pair  $(x, y)$  from the complete relation  $U (U : M \times M)$  only one relation takes place, either  $xAy$  or  $x\bar{A}y$ , but not both of them simultaneously.

For relations we can define a notion of an inclusion. We shall write  $A \subseteq B$  if the set of pairs for which  $A$  is true is included in the set of pairs for which  $B$  is true. As a result, we shall write  $A \subset B$  if the set  $A$  of pairs is a subset of the set  $B$ , where  $A \neq B$ .

It is important to note the following property of an inclusion: if  $A \subseteq B$ , then  $xAy$  implies that  $xBy$ . And conversely, if  $xAy$  implies  $xBy$ , then  $A \subseteq B$ .

This implies that for any relation  $A$ , we have  $\emptyset \subseteq A \subseteq U$ , where  $\emptyset$  is the empty relation and  $U = M \times M$  the complete relation.

Now let us introduce some operations that are not immediately reduced to those with sets. The simplest one is the transition to the inverse relation. If  $A$  is a relation on the set  $M$ , then the inverse relation  $A^{-1}$  is defined by  $xA^{-1}y = yAx$ .

An important role is played by the operation of the product of two relations, denoted by  $AB$ . This operation is defined as follows: to say that  $xABy$  is true means that there is  $z \in M$  such that  $xAz$  and  $zBy$  are true.

We define another important operation called the transitive closure of the relation  $A$  and denoted by  $\hat{A}$ . This operation is defined as follows. The relation  $x\hat{A}y$  is true if there is a chain of elements  $z_0, z_1, \dots, z_k = y$  from  $M$  such that  $z_0Az_1, z_1Az_2, \dots, z_{k-1}Az_k$ .

It is easy to show that

$$\hat{A} = A \cup A^2 \cup A^3 \cup \dots \cup A^n. \quad (\text{A.7})$$

Since the operations of intersection and union of relations came from their analogues defined on sets, their properties are exactly the same.

Let us recall some algebraic properties of other operations.

$$(1) \quad (A^{-1})^{-1} = A; \quad (\text{A.8})$$

(2) the diagonal relation  $E(xEy \Leftrightarrow x = y)$  plays the role of the identity:

$$AE = EA = A; \quad (\text{A.9})$$

$$(3) \quad A\emptyset = \emptyset A = \emptyset; \quad (\text{A.10})$$

(4) there is an associative law for the product of relations:

$$(AB)C = A(BC);$$

$$(5) \quad (AB)^{-1} = B^{-1}A^{-1};$$

(6) If the relation  $A$  is surjective, then

$$A^{-1}A \supseteq E;$$

(7) there is a distributive law of multiplication with respect to union:

$$(A \cup B)C = (AC) \cup (BC);$$

(8) there is a distributive law of multiplication with respect to intersection:

$$(A \cap B)C \subseteq (AC) \cup (BC);$$

(9)

$$(A \cup B)^{-1} = A^{-1} \cup B^{-1};$$

(10)

$$(A \cap B)^{-1} = A^{-1} \cap B^{-1};$$

(11) if  $A \subseteq B$  then  $\hat{A} \subseteq \hat{B}$ ;

(12) if  $A \subseteq B$  then  $A^{-1} \subseteq B^{-1}$ ;

(13) if  $A \subseteq B$  then  $AC \subseteq BC$  and  $CA \subseteq CB$ ;

(14)  $\hat{\hat{A}} = A$ .

Let us define some operations using the basic operations.

(1) symmetrizable product:

$$A \circ B = AB \cup BA;$$

(2) transitive closure of a union:

$$A \hat{\cup} B = A \hat{\cup} B;$$

(3) transitive closure of the symmetrizable product:

$$A \hat{\circ} B = A \hat{\circ} B.$$

From this definition it follows that these three operations are commutative. Note that the associative law for the symmetrizable product does not have to hold for the general case.

### A.1.5 Properties of Relations

**Definition A.9.** A relation  $A$  is called *reflexive* if  $E \subseteq U$ . In other words, a reflexive relation between an object and itself is always true:  $xAx$ .

**Definition A.10.** A relation  $A$  is called *antireflexive* if  $xAy$  implies  $x \neq y$ , that is, in algebraic notation,  $A \cap E = \emptyset$ . In other words,  $A \not\subseteq E$ , that is, a relation  $A$  can be true only for distinct objects.

**Theorem A.4.** A relation  $A$  is antireflexive if and only if  $x\bar{A}x$  for any  $x \in M$ .

**Definition A.11.** A relation  $A$  is called *symmetric* if  $A \subseteq A^{-1}$ , in other words, if  $xAy$  implies  $yAx$ .

**Theorem A.5.** *A relation  $A$  is symmetric if and only if  $A = A^{-1}$*

The proof follows immediately from the monotonicity of the operation of inversion.

**Definition A.12.** A relation  $A$  is called *asymmetric* if  $A \cap A^{-1} = \emptyset$ . This means that  $xAy$  or  $yAx$  is false.

**Theorem A.6.** *If a relation  $A$  is asymmetric, then it is antireflexive.*

**Definition A.13.** A relation  $A$  is called *antisymmetric* if  $A \cap A^{-1} \subseteq E$ . This means that both  $xAy$  and  $yAx$  are simultaneously true if and only if  $x = y$ .

**Definition A.14.** A relation  $A$  is called *transitive* if  $A^2 \subseteq A$ .

From this we conclude that if  $xAz$  and  $zAy$  are true, then  $xAy$  is also true. By induction, we can conclude that if  $xAz_1$  and  $z_1Az_2, \dots, z_{n-1}Ay$  are true, then  $xAy$ .

*Remark A.1.* It is easy to show that if a reflexive relation  $A$  is transitive, then  $A^2 = A$ .

**Theorem A.7.** *If a relation  $A$  is transitive, then  $A = \hat{A}$ , and conversely, if  $A = \hat{A}$  then  $A$  is transitive.*

**Theorem A.8.** *For any relation  $A$ , its transitive closure  $\hat{A}$  is equal to the intersection  $\cap B$  of all transitive relations  $B$  that include  $A$ .*

*Remark A.2.* If a relation  $(A, M)$  is a contraction of the relation  $(A_1, M_1)$ , it automatically possesses all the properties of the given relation.

**Theorem A.9.** *If relations  $A$  and  $B$  are reflexive, then the following relations are reflexive too:*

$$A \cup B, \quad A \cap B, \quad A^{-1}, \quad AB, \quad \hat{A}.$$

**Theorem A.10.** *If relations  $A$  and  $B$  are antireflexive, then the following relations are antireflexive too:*

$$A \cup B, \quad A \cap B, \quad A^{-1}.$$

**Theorem A.11.** *The product  $AB$  of two antireflexive relations is antireflexive if and only if  $A \cap B^{-1} = \emptyset$ .*

**Theorem A.12.** *If relations  $A$  and  $B$  are symmetric, then the following relations are symmetric too:*

$$A \cup B, \quad A \cap B, \quad A^{-1}.$$

**Theorem A.13.** *The product  $AB$  of two symmetric relations  $A$  and  $B$  is symmetric if and only if  $A$  and  $B$  commute, that is,  $AB = BA$ .*

**Corollary A.1.** *The transitive closure  $\hat{A}$  of a symmetric relation  $A$  is a symmetric relation.*

**Theorem A.14.** (1) *If relation  $A$  is asymmetric, its intersection  $A \cap B$  with an arbitrary relation  $B$  is asymmetric.*

(2) *If relation  $A$  is a symmetric, then the relation  $A^{-1}$  is a symmetric too.*

**Theorem A.15.** *The union  $A \cup B$  of asymmetric relations  $A$  and  $B$  is asymmetric if and only if  $A \cap B^{-1} = \emptyset$ .*

**Theorem A.16.** *If relations  $A$  and  $B$  are antisymmetric, then the following relations are antisymmetric too:*

$$A \cap B, \quad A^{-1}.$$

**Theorem A.17.** *The union  $A \cup B$  of two antisymmetric relations  $A$  and  $B$  is antisymmetric if and only if  $A \cap B^{-1} \subseteq E$  is true.*

**Theorem A.18.** *If relations  $A$  and  $B$  are transitive, then the following relations are transitive also:*

$$A \cap B, \quad A^{-1}, \quad \hat{A}.$$

**Theorem A.19.** *An antireflexive and transitive relation is asymmetric.*

**Definition A.15.** A relation  $A$  is called *negatively reflexive* if a relation  $\bar{A}$  is transitive, that is, in algebraic notation,  $A^2 \subseteq A$ .

**Theorem A.20.** *A relation  $A$  is negatively transitive if and only if for all  $x, y, z \in M$ , the relation  $xAy$  implies  $xAz$  or  $zAy$ .*

**Theorem A.21.** *A symmetric and negatively transitive relation is transitive.*

For convenience, we introduce the following notation:

$$A^0 = A \cap A^{-1}, \quad A' = A \setminus A^{-1}.$$

It is clear that  $A' = A \setminus A^0$ .

**Definition A.16.** A relation  $A$  is called *coherent* if  $A \cup A^{-1} = M \times M$ . This means that for all  $x, y \in M$ , either  $xAy$  or  $yAx$  is true.

**Definition A.17.** A relation  $A$  is called *weakly coherent* if  $A \cup A^{-1} \cup E = M \times M$ . This means that for all  $x, y \in M$ ,  $x \neq y$ , either  $xAy$  or  $yAx$  is true.

We consider now some types of relations.

### A.1.6 Equivalence Relations

**Definition A.18.** A relation  $A$  on set  $M$  is called an *equivalence relation* if it is reflexive, symmetric, and transitive.

**Definition A.19.** The set of the elements equivalent to  $x$  is called the *equivalence class* of the element  $x$  and is denoted by  $M_x$ .

**Definition A.20.** A disjoint system of nonempty subsets  $\{M_\alpha, \alpha \in A\}$  is called a *partition* (or a subdivision) of the set  $M$  if  $M = \bigcup_{\alpha \in A} M_\alpha$ . The sets  $M_\alpha$  themselves are called *classes* of the given partition.

**Theorem A.22.** Any relation  $A$  on the set  $M$  partitions (divides) the set  $M$  on equivalence classes, that is,

(1)

$$\bigcup_{x \in M} M_x = M;$$

(2)

$$M_x \neq M_y \implies M_x \cap M_y = \emptyset.$$

Conversely, for any partition  $\{M_i : i \in I\}$  of the set  $M$ , the binary relation  $A$  defined as “belongs to the same class,” that is,

$$xAy \Leftrightarrow (x, y) \in M_i, \quad i \in I,$$

is an equivalence relation.

**Definition A.21.** The set of all equivalence classes of the set  $M$  with respect to the binary relation  $A$  is called a *factor set* of the set  $M$  with respect to the relation  $A$  and is denoted by  $M/A$ .

**Definition A.22.** The set  $M_A$  is called a *set of representatives* of the factor set  $M/A$  if it possesses the following properties:

(1)  $M_A \subset M$ ;(2)  $x, y \in M_A \implies x\bar{A}y$ ;(3)  $\bigcup_{x \in M_A} M_x = M$ .

**Theorem A.23.** Any equivalence relation  $A$  on the set  $M$  can be represented algebraically as

$$A = (M_A^*)^{-1} M_A^*,$$

where  $M_A^*$  is a simpler relation on the set  $M$ :

$$xM_A^*y \Leftrightarrow \begin{cases} x \in M_A; \\ y \in M_x. \end{cases}$$

**Theorem A.24.** *Let  $M$  be a finite set and  $A$  an equivalence relation on  $M$ . Then there exist  $n$  and  $m$  such that to any  $x \in M$  one can associate an ordered collection of  $n + m$  binary indicators (ones and zeros):*

$$\begin{aligned} x &\rightarrow \langle \xi_1, \xi_2, \dots, \xi_n; \xi_{n+1}, \dots, \xi_{n+m} \rangle, \\ y &\rightarrow \langle \eta_1, \eta_2, \dots, \eta_n; \eta_{n+1}, \dots, \eta_{n+m} \rangle, \end{aligned}$$

*etc., in such a way that*

- (1) *distinct ordered collections correspond to different elements;*
- (2)  *$xAy$  is true if and only if the first  $n$  indicators of these elements coincide:*  
 $\xi_1 = \eta_1, \xi_2 = \eta_2, \dots, \xi_n = \eta_n.$

This theorem shows that any equivalence relation (on a finite set) can be represented as a coincidence of a certain collection of some indicators.

**Definition A.23.** A *direct sum* of relations  $\langle A_1, M_1 \rangle$  and  $\langle A_2, M_2 \rangle$  is a relation  $\langle A_1 \cup A_2, M_1 \cup M_2 \rangle$  denoted by  $\langle A_1, M_1 \rangle \oplus \langle A_2, M_2 \rangle$ .

**Theorem A.25.** *If  $M_1 \cap M_2 = \emptyset$  and relations  $A_1$  and  $A_2$  are equivalence relations, then their direct sum  $\langle A, M \rangle = \langle A_1, M_1 \rangle \oplus \langle A_2, M_2 \rangle$  is also an equivalence relation.*

**Theorem A.26.** *If relations  $A$  and  $B$  are equivalence relations, then relation  $A \cap B$  is an equivalence relation also.*

**Definition A.24.** Relations  $A$  and  $B$  are called *coherent* if the set  $M$  can be represented as a union of disjoint subsets  $M_1$  and  $M_2$  (one of which can be empty) such that

$$\begin{aligned} \langle A, M \rangle &= \langle A_1, M_1 \rangle \oplus \langle A_2, M_2 \rangle, \\ \langle B, M \rangle &= \langle B_1, M_1 \rangle \oplus \langle B_2, M_2 \rangle, \end{aligned}$$

and

$$A_1 \subseteq B_1, \quad B_2 \subseteq A_2.$$

Let us formulate as theorems some properties of the simplest operations on equivalence relations.

**Theorem A.27.** *The union  $A \cup B$  of the equivalence relations  $A$  and  $B$  is an equivalence relation if and only if relations  $A$  and  $B$  are coherent.*

**Theorem A.28.** *The union  $A \cup B$  of two equivalence relations is an equivalence relation if and only if  $AB = A \cup B$ .*

**Theorem A.29.** *In order that the product  $AB$  of equivalence relations  $A$  and  $B$  be an equivalence relation, it is necessary and sufficient that  $A$  and  $B$  commute, that is  $AB = BA$ .*

**Theorem A.30.** *If relations  $A$  and  $B$  are equivalence relations, then relations  $A\hat{\cup}B$  and  $A\hat{\delta}B$  are equivalence relations also.*

It turns out that the operation  $A\hat{\cup}B$  (it is sometimes called the union of equivalence relations) is associative.

**Theorem A.31.** *For any transitive relations  $A, B$  and  $C$ , the following associative law is true:*

$$(A\hat{\cup}B)\hat{\cup}C = A\hat{\cup}(B\hat{\cup}C).$$

**Theorem A.32.** *If  $A$  and  $B$  are equivalence relations, then*

$$A\hat{\cup}B = A\hat{\delta}B.$$

**Theorem A.33.** *If  $A$  and  $B$  commute, then*

$$AB = A\hat{\cup}B.$$

These results can be interpreted algebraically. The set  $\mathcal{M}$  of all relations on  $M$  possesses the structure of a monoid (semigroup with unit) with respect to the operation of the product of relations. Let the set  $\mathcal{M}_{\supseteq} \subseteq \mathcal{M}$  consist of equivalence relations. Any subset of the set  $\mathcal{M}_{\supseteq}$  closed with respect to the operation of product  $AB$  is a commutative monoid (A.29). Since for a set  $M$  that includes at least three elements there exist noncommutative equivalence relations, the set  $\mathcal{M}_{\supseteq}$  itself does not form a monoid with respect to the product of relations. However,  $\mathcal{M}_{\supseteq}$  possesses the structure of a commutative monoid with respect to the operation  $A\hat{\cup}B$  (A.31) or  $A\hat{\delta}B$  (A.32). The operation  $A\hat{\cup}B$  on submonoids from  $\mathcal{M}_{\supseteq}$  of the monoid  $\mathcal{M}$  (with respect to  $AB$ ) coincides with the operation of the product  $AB$  (A.33).

### A.1.7 Strict Order Relations

**Definition A.25.** A relation  $A$  on the set  $M$  is called a *strict (partial) order relation* if it is antireflexive and transitive.

Sometimes, if it does not lead to misunderstanding, a strict partial order is called simply a strict order.

**Theorem A.34.** *If a relation  $A$  is a strict order relation, it is asymmetric.*

The set  $M$  along with the strict order relation  $A$  on  $M$ , that is, the pair  $\langle M, A \rangle$ , is called an ordered set.

**Definition A.26.** A strict order relation  $A$  is called a *perfect strict order* if  $A$  is weakly coherent (connected).

**Definition A.27.** The element  $x \in M$  is called the *smallest* element in an ordered set  $\langle M, A \rangle$  if for any  $y \in M, x \neq y$ , the relation  $xAy$  is true.

We now describe the structure of finite sets with a perfect strict order on them.

**Theorem A.35.** *If on a finite (nonempty) set  $M$  there is a perfect strict order  $\prec$ , then the smallest element in the ordered set  $\langle M, \prec \rangle$  is unique.*

We observe that if there is a perfect strict order on  $M$ , then on any nonempty subset  $Q$  of the set  $M$  the perfect strict order emerges naturally, and thus on the basis of Theorem A.35, if  $Q$  is finite, the smallest element on  $Q$  is unique.

**Theorem A.36.** *Let a perfect strict order relation  $\prec$  be given on a set  $M$ . Then one can choose an enumeration  $M = \{x_1, x_2, \dots, x_n\}$  such that the relation  $x_i \prec x_j$  is true if and only if  $i < j$ .*

This theorem implies that any perfect strict order on a finite set  $M$  is equivalent to the order on a certain segment of the natural numbers.

If the order on the set  $M$  is not perfect, then obviously, one cannot enumerate the elements of this set in such a way that the largest numbers correspond to the largest elements.

**Definition A.28.** The element  $x \in M$  is called *minimal (maximal)* in the ordered set  $\langle M, \prec \rangle$  if there is no element  $y$  such that  $y \prec x$  ( $x \prec y$ ).

We remark that in the case of a perfect strict order, the minimal element coincides with the smallest one. In the general case, the smallest element is the minimal element, but not necessarily conversely.

**Definition A.29.** The elements  $x$  and  $y$  are called *comparable* in a given ordered set  $\langle M, \prec \rangle$  if  $x \prec y$  or  $x = y$  or  $y \prec x$ .

**Definition A.30.** A subset  $Q \subseteq M$  is called the *maximal perfect subset* in the ordered set  $\langle M, \prec \rangle$  if:

- (1) the relation  $\prec$  establishes on  $Q$  a perfect strict order;
- (2) on any subset  $P$  of the set  $M$  such that  $P \subset Q$ , the relation  $\prec$  is not a perfect strict order.

**Theorem A.37.** *Let  $\langle M, \prec \rangle$  be an ordered set. For any element  $y \in M$ , there is a maximal perfect subset  $Q$  of the set  $M$  that includes  $y$ .*

Suppose now that on the set  $M$  we have a strict order relation  $\prec$  and a certain topology.

**Definition A.31.** The element  $x \in M$  is called the *upper (lower) boundary* of the set  $Q \subseteq M$  if for any element  $y \in Q$  either  $x \prec y$  or  $x = y$  is true (either  $y \prec x$  or  $y = x$  is true).

**Definition A.32.** The order  $\prec$  is called *continuous with respect to a given topology* if any lower (upper) boundary of an arbitrary  $Q \subseteq M$  is the upper (lower) boundary of its closure  $\bar{Q}$  (in the given topology).

It would probably be more natural to define the continuity of an order relation by means of the condition that the union of the graph and the diagonal be closed on  $M \times M$ . It is easy to show that this definition implies the previous one.

**Theorem A.38.** *If an order is continuous, then the set  $R_x$  of all elements  $y \in M$  such that either  $y \prec x$  or  $x = y$  is closed.*

**Theorem A.39.** *If an order is continuous, then any maximal perfect set  $Q$  is closed.*

**Theorem A.40.** *Let  $M$  be a compact topological space and  $\prec$  a continuous order on it. Then for any element  $y \in M$  there exists a minimal element  $x_0$  such that either  $x_0 \prec y$  or  $x_0 = y$ .*

**Theorem A.41.** *Let  $M$  be a topological space and  $\prec$  a continuous order on it. Then if the set  $R_x$  of all elements  $y \in M$  such that either  $x \prec y$  or  $x = y$  is compact, then for any  $y \in M$  there exists a minimal element  $x_0$  such that either  $x_0 \prec y$  or  $x_0 = y$ .*

### A.1.8 Nonstrict Order Relations

**Definition A.33.** A relation  $A$  on the set  $M$  is called a *nonstrict order relation* if it can be represented as

$$A = A_1 \cup E,$$

where  $A_1$  is a strict order on  $M$  and  $E$  is a diagonal relation.

**Theorem A.42.** *A relation  $A$  is a nonstrict order if and only if it is reflexive, antisymmetric, and transitive.*

**Definition A.34.** A nonstrict order  $A$  is called *perfect* if  $A$  is connected. A perfect nonstrict order is called a *linear order*, while an imperfect nonstrict order is called a *partial order*.

**Theorem A.43.** *If  $A$  is a linear order, then  $A_1 = A \setminus E$  is a perfect strict order. Conversely, if  $A_1$  is a perfect strict order, then  $A = A_1 \cup E$  is a linear order.*

**Definition A.35.** A relation  $A$  on a set  $M$  is called a *quasiorder* (or *preorder*) *relation* if it is reflexive and transitive.

Obviously, a preorder (quasiorder) is a generalization of an equivalence relation, and simultaneously a generalization of a nonstrict order: a symmetric quasiorder is an equivalence relation, and a symmetric quasiorder is a nonstrict order.

**Theorem A.44.** *If a relation  $A$  is an equivalence relation and a nonstrict order relation simultaneously, then it is the relation of equality.*

**Theorem A.45.** *If  $A$  is a quasiorder, then the relation  $B = A \cap A^{-1}$  is an equivalence relation.*

Let  $A$  be a quasiorder on a set  $M$ . Denote by  $\mathcal{M}$  the collection of equivalence classes with respect to the relation  $B = A \cap A^{-1}$ . We say that two classes  $X$  and  $Y$  from  $\mathcal{M}$  are in relation  $A^*$  if one can find in both classes representatives  $x \in X$ ,  $y \in Y$  such that  $xAy$  is true. We say in this case that the relation  $A^*$  is induced by the quasiorder  $A$ .

**Theorem A.46.** *The relation  $A^*$  on the set of equivalence classes  $\mathcal{M}$  induced by the quasiorder  $A$  is a nonstrict order.*

So if there is a quasiorder on  $M$ , one can construct a nonstrict order by means of “gluing” different objects from  $M$ .

We now state as theorems the simplest properties of operations on order relations.

**Theorem A.47.** *If the relation  $A$  is a strict order (nonstrict order, quasiorder), then the relation  $A^{-1}$  is a strict order (nonstrict order, quasiorder).*

**Theorem A.48.** *If the relation  $A$  is a perfect strict order (nonstrict order, quasiorder), then the relation  $A^{-1}$  is a perfect strict order (perfect nonstrict order, perfect quasiorder).*

**Theorem A.49.** *If  $A$  and  $B$  are strict orders (nonstrict orders, quasiorders), then the intersection  $A \cap B$  is also a strict order (nonstrict order, quasiorder).*

**Corollary A.2.** *The intersection of a strict and a nonstrict order is a strict order.*

**Theorem A.50.** *If  $A$  and  $B$  are strict orders, then their union  $A \cup B$  is a strict order if and only if  $BA \cup AB = A \cup B$ .*

**Theorem A.51.** *The union  $A \cup B$  of nonstrict orders  $A$  and  $B$  is a nonstrict order if and only if*

$$BA \cup AB \subseteq A \cup B, \quad A \cap B^{-1} \subseteq E.$$

We consider one more operation that for orders is, in a certain sense, the inverse to the operation of transitive closure.

**Definition A.36.** The *reduction* of a relation  $A$  is the relation  $A^R$  defined by the condition  $A^R = A \setminus A^2$ .

This means that  $xA^Ry$  is true if and only if  $xAy$  itself is true but there does not exist an “intermediary”  $z$  such that  $xAz$  and  $zAy$  are true. In other words, the relation  $xA^Ry$  means “immediate subordination” of the element  $x$  to the element  $y$ .

**Theorem A.52.** For any relation  $A$ ,

$$(\hat{A})^R \subseteq A.$$

**Theorem A.53.** If  $A$  is a strict order on a finite set  $M$ , then  $\hat{A}^R = A$ .

This theorem states that a strict order is restorable from its reduction. This is not so in the general case. An example is any reflexive relation (if  $A$  is reflexive, then  $A^R = \emptyset$ ).

*Remark A.3.* Unfortunately, Theorem A.53 is not valid on infinite sets. For example, let  $A$  be the natural order  $<$  on the set of real numbers. Then  $A^R = \emptyset$ , which means that  $\hat{A}^R \neq \emptyset \neq A$ .

Moreover, the reduction  $A^R$  is the minimal relation that permits the restoration of  $A$ . The following theorem explains the exact meaning of this statement.

**Theorem A.54.** If the relation  $B$  is such that  $\hat{B} = A$ , then  $A^R \subseteq B$ .

Theorem A.53 implies that if  $A$  is a strict order on a finite set and  $xAy$  is true, then there is a minimal number  $n$  such that  $x(A^R)^ny$  is true. This  $n$  characterizes the length of the minimal path on the graph of the relation  $A^R$  that connects  $x$  and  $y$ .

We now formulate as theorems some properties of reductions of strict orders.

**Definition A.37.** A relation  $A$  is called *antitransitive* if for all  $n \geq 2$ , we have  $A \cap A^n = \emptyset$ .

In other words, if the chain of relations  $xAx_1, x_1Ax_2, \dots, x_nAy$  is true, then  $xAy$  is impossible. This means that the path on the graph of the relation  $A$  between the vertices  $x$  and  $y$  excludes a cycle.

**Theorem A.55.** Any antitransitive relation is asymmetric and thus antireflexive.

**Theorem A.56.** If  $A$  is a strict order, then the relation  $A^R$  is antitransitive.

The converse holds as well:

**Theorem A.57.** If the relation  $B$  is antitransitive, then  $B$  is a reduction of a certain strict order.

### A.1.9 Preference Relations

**Definition A.38.** A binary relation  $R$  on a set  $Y$  is called a *weak ordering* if  $R$  on  $A$  is asymmetric and negatively transitive.

**Theorem A.58.** A binary relation is a strict order if and only if it is weakly connected and weakly ordered.

Take a weak ordering as a strict order relation and denote it by the symbol  $\prec$  ( $x \prec y$  reads “ $x$  is (strictly) less preferable to  $y$ ” or “ $y$  is (strictly) preferable to  $x$ ”). We can define a relation of indifference  $\sim$  as an absence of strict preference:

$$x \sim y \Leftrightarrow (x \prec y) \quad \text{or} \quad x \succ y$$

The asymmetry is an “obvious” condition of a preference relation. It can be considered a criterion of this relation. The transitivity follows from asymmetry and negative transitivity (Theorem A.20) and seems to be a reasonable criterion of the consistency of individual preferences. However, the concept of weak ordering provides an individual with an unlimited possibility of judgment about a preference, as follows from (A.19).

Let us define the nonstrict preference relation  $\preceq$  as the union of relations  $\prec$  and  $\sim$ :

$$x \preceq y \Leftrightarrow x \prec y \quad \text{or} \quad x \sim y.$$

**Theorem A.59.** Suppose that the relation  $\prec$  is a weak ordering on  $X$ , that is, it is asymmetric and negatively transitive. Then

(1) For all  $x, y \in X$  only one of the three following relations holds:

- (i)  $x \prec y$ ,
- (ii)  $y \prec x$ ,
- (iii)  $x \sim y$ .

(2) The relation  $\prec$  is transitive.

(3) The relation  $\sim$  is an equivalence relation.

(4)

$$(x \prec y, y \sim z) \Rightarrow x \prec z, \quad (x \sim y, y \prec z) \Rightarrow x \prec z;$$

(5) The relation  $\preceq$  is transitive and connected.

(6) If the relation  $\prec'$  on  $X/\sim$  is defined by the relation  $a \prec' b$  (that is, if there exist  $x \in a$  and  $y \in b$  such that  $x \prec y$ ), then  $\prec'$  on  $X/\sim$  is a perfect strict order.

**Theorem A.60.** If the relation  $\prec$  on  $X$  is a weak ordering and the set  $X/\sim$  is countable, then there exists a real function  $u$  on  $X$  for which

$$x \prec y \Leftrightarrow u(x) < u(y)$$

is true for any  $x, y \in X$ .

Such a function is called a utility function that preserves ordering.

**Theorem A.61.** *If (A.60) is true, then the relation  $\prec$  on  $X$  must be a relation of weak ordering.*

Hence if  $\prec$  on  $X$  is not a relation of weak order, then it is not possible to ensure the validity of (A.60), regardless of the capacity of the set  $X$ .

**Definition A.39.** A binary relation  $R$  on the set  $X$  is called a *strict partial ordering* if it is antireflexive and transitive.

Since a strict partial ordering  $\prec$  on  $X$  admits the relations  $x \sim y, y \sim z, x \prec z$ , the relation  $\sim$  does not have to be transitive and therefore does not have to be an equivalence relation. However, the relation  $\approx$  defined as

$$x \approx y \Leftrightarrow (x \sim z \Leftrightarrow y \sim z) \quad \text{for all } z \in X$$

turns out to be transitive.

**Theorem A.62.** *If the relation  $\prec$  on  $X$  is a strict partial ordering, then*

(1) *For any  $x, y \in X$  only one of the four following relations is true:*

- (i)  $x \prec y$ ,
- (ii)  $y \prec x$ ,
- (iii)  $x \approx y$ ,
- (iv)  $(x \sim y, x \approx y)$ .

(2) *The relation  $\approx$  is an equivalence relation.*

(3)

$$x \approx y \Leftrightarrow (x \prec z \Leftrightarrow y \prec z, z \prec x \Leftrightarrow z \prec y \forall z \in X);$$

(4) (i)  $(x \prec y, y \approx z) \implies (x \prec z)$ ,

(ii)  $(x \approx y, y \prec z) \Leftarrow (x \prec z)$

(5) *Define the relation  $\prec^*$  on  $X/\approx$  as “ $a \prec^* b \Leftrightarrow$  there exist  $x \in a$  and  $y \in b$  such that  $x \prec y$ .” In this case the relation  $\prec^*$  on  $X/\approx$  is not a strict partial order.*

**Theorem A.63.** *If the relation  $\prec^*$  is a strict partial ordering on  $X$ , then one can find on  $X$  a strict ordering  $\prec^0$  that contains the relation  $\prec^*$ , that is,*

$$x \prec^* y \implies \prec^0 \quad \text{for all } x, y \in X.$$

**Theorem A.64.** *If the relation  $\prec$  on  $X$  is a strict partial ordering and the set  $X/\approx$  is countable, then there exists a real function  $u$  on  $X$  such that for any  $x, y \in X$ , the following relations hold:*

$$x \prec y \implies u(x) < u(y), \quad x \approx y \implies u(x) = u(y).$$

We shall call an ordering used to describe the orientation of an individual with respect to a goal a *preference*. This notion will be complemented if necessary by the same elements as an ordering: strict, nonstrict, partial, etc.

**Definition A.40.** A real function  $u$  on the ordered set  $X$  is called a *utility function* if it is monotone, that is,

$$u(x) > u(y) \Leftrightarrow x \succeq y \quad \text{and} \quad u(x) \leq u(y) \Leftrightarrow x \preceq y \quad (\text{A.11})$$

for all  $x, y \in X$ .

For a strictly ordered set both of the above relations are replaced by the strict ones  $<$ ,  $\prec$ . Other combinations of relations are used. For example,

$$u(x) < u(y) \text{ if } x \prec y, \quad u(x) = u(y) \text{ if } x \sim y.$$

**Definition A.41.** A subset  $A$  of the set  $X$  is called *dense with respect to the ordering*  $\prec$  in  $X$  if for any  $x, y \in X \setminus A$ ,  $x \prec y$ , there is  $z \in A$  such that  $x \prec z$ .

**Theorem A.65.** On a strictly ordered infinite set, a utility function exists if and only if it contains a countable set dense with respect to the ordering.

If instead of a strict ordering there is a nonstrict one, then the above theorem should be applied to the set  $X/\sim$ . Furthermore, assuming a linear structure of the set  $X$ , it is possible to question the existence of a linear utility function.

Consider a nonstrict ordering  $\succcurlyeq$  on a convex set  $X$ . Suppose that the following two conditions are satisfied:

**Condition 5** For any  $x, y, z \in X$  the sets

$$\{\alpha : \alpha x + (1 - \alpha)y \succcurlyeq z\}, \quad \{\alpha : z \succcurlyeq \alpha x + (1 - \alpha)y\},$$

from the interval  $[0, 1]$  are closed.

**Condition 6** For any  $x, y, z \in X$ ,  $x \sim y$ , and any real number  $\alpha$ ,  $0 \leq \alpha \leq 1$ ,

$$\alpha x + (1 - \alpha)z \sim \alpha y + (1 - \alpha)z.$$

**Theorem A.66.** A nonstrict ordering  $\succcurlyeq$  on a convex set  $X$  satisfies Conditions 5 and 6 if and only if there is a linear utility function (Theorem A.40) unique with respect to an increasing linear transformation.

Note that other variants of necessary and sufficient conditions similar to Conditions 5 and 6 are possible.

One of the most studied cases of a linear utility function is an expected utility, used in the case that the alternatives are random values or events.

For any alternative  $x$  with a random outcome one can find a correspondence with a certain probability measure and therefore consider that the set  $X$  consists of probability measures defined on a certain  $\sigma$ -algebra over the set of values of random variables  $S$ .

**Condition 7** Let

$$x(B) = 1, \quad B \subset S, \quad B = \cup_{i=1}^{\infty} B_i, \quad B_i \cup B_j = \emptyset, \quad i \neq j.$$

Then for all  $S \in B_i, i = 1, 2, \dots,$

$$\begin{aligned} x \succcurlyeq \sum_i x(B_i)S_i, & \quad \text{if } S \succcurlyeq S_i, \\ x \preccurlyeq \sum_i x(B_i)S_i, & \quad \text{if } S \preccurlyeq S_i. \end{aligned}$$

**Theorem A.67.** Let  $X$  be the set of all probability measures over the  $\sigma$ -algebra of outcomes  $S$  containing all one-point subsets of  $S$  and such that the ordering  $\succcurlyeq$  on it satisfies Conditions 5, 6, 7. Then there exists a utility function of the form

$$u(x) = \int_S u(s)dX(S).$$

Conditions 5, 6, 7 also are necessary conditions.

### A.1.10 Mappings of Relations

Consider two relations  $(A, M)$  and  $(B, L)$ . The notation  $\alpha : (A, M) \rightarrow (B, L)$  will signify that  $\alpha$  is a mapping of the set  $M$  into the set  $L$ .

**Definition A.42.** A mapping  $\alpha : M \rightarrow L$  is called a *homomorphic mapping* or a *homomorphism* of the relation  $(A, M)$  into the relation  $(B, L)$  if  $xAx'$  implies  $\alpha(x)B\alpha(x')$ .

If a homomorphism  $\alpha$  is surjective (injective, bijective), then it is called an *epimorphism* (*monomorphism*, *isomorphism*).

**Definition A.43.** The mapping  $\alpha : M \rightarrow L$  is called a *correspondence* of the relation  $(A, M)$  into the relation  $(B, L)$  if  $\alpha(x)B\alpha(x')$  implies  $xAx'$ . A homomorphism (epimorphism, monomorphism, isomorphism)  $\alpha$  is called a *K-homomorphism* (*K-epimorphism*, *K-monomorphism*, *K-isomorphism*) if  $\alpha$  is simultaneously a correspondence.

**Theorem A.68.** If  $(A, M)$  is a quasiorder, then there is a *K-epimorphism*  $\alpha : (A, M) \rightarrow (B, L)$  such that  $B$  is a nonstrict order and  $L = M/A \cap A^{-1}$ .

Any homomorphism can be extended to a *K-homomorphism*:

**Theorem A.69.** Let  $\alpha : M \rightarrow L$  be a mapping, and  $(C, L)$  a relation. Then

- (1) there is a unique relation  $(D, M)$  such that  $\alpha$  is a  $K$ -homomorphism of the relation  $(D, M)$  into the relation  $(C, L)$ ;
- (2) for any relation  $(B, M)$  for which  $\alpha$  is a homomorphism of the relation  $(B, M)$  into the relation  $(C, L)$ , we have  $B \subseteq D$ .

We consider now which properties of relations are preserved for different mappings.

**Theorem A.70.** Let  $\alpha : (A, M) \rightarrow (B, L)$  be a homomorphism. Then

- (1) if  $\alpha$  is surjective and  $A$  is reflexive, then  $B$  is reflexive;
- (2) if  $B$  is antireflexive, then  $A$  is antireflexive.

**Theorem A.71.** Let  $\alpha : (A, M) \rightarrow (B, L)$  be a correspondence. Then

- (1) the reflexivity of the relation  $B$  implies the reflexivity of the relation  $A$ ;
- (2) if  $\alpha$  is surjective, then the antireflexivity of  $A$  implies the antireflexivity of  $B$ .

**Theorem A.72.** If  $\alpha : (A, M) \rightarrow (B, L)$  is a  $K$ -epimorphism, then  $B$  is symmetric if and only if  $A$  is symmetric.

**Theorem A.73.** If  $\alpha : (A, M) \rightarrow (B, L)$  is a  $K$ -isomorphism, then  $B$  is antisymmetric if and only if  $A$  is antisymmetric.

**Theorem A.74.** If  $\alpha : (A, M) \rightarrow (B, L)$  is a  $K$ -epimorphism, then  $B$  is transitive if and only if  $A$  is transitive.

**Theorem A.75.** If  $\alpha : (A, M) \rightarrow (B, L)$  is a  $K$ -epimorphism, then relation  $B$  is an equivalence (quasiorder) if and only if  $A$  is an equivalence relation (quasiorder).

**Theorem A.76.** Let the mapping  $\alpha$  of the set  $M$  to the set  $L$  be a homomorphism (correspondence) of the relation  $(A, M)$  into the relation  $(B, L)$  and simultaneously let  $\alpha$  be a homomorphism (correspondence) of the relation  $(A_1, M)$  into the relation  $(B_1, L)$ . Then the mapping  $\alpha$  is also a homomorphism (correspondence) of the relations  $(A \cup A_0, M)$ ,  $(A \cap A_1, M)$ , and  $(AA_1, M)$  into the relations  $(B \cup B_1, L)$ ,  $(B \cap B_1, L)$ , and  $(BB_1, L)$  respectively.

## A.2 Elements of the Theory of Real Functions

### A.2.1 Rings, Semirings, and Algebras of Sets

**Definition A.44.** Let  $M$  be an arbitrary set. A nonempty collection  $\mathfrak{U}$  of some subsets of  $M$  is called a *ring* (of sets) if  $A, B \in \mathfrak{U}$  implies that

1.

$$A \cup B \in \mathfrak{U};$$

2.

$$A \setminus B \in \mathfrak{U}.$$

Hence a ring is a collection of sets closed under union and difference of a finite number of sets. Every ring contains the empty set and is closed under intersection of a finite number of sets. Indeed, let  $A \in \mathfrak{U}$  (such an  $A$  exists as long as  $\mathfrak{U}$  is nonempty). Then  $\emptyset = A \setminus A$  and therefore  $A \in \mathfrak{U}$ . Let  $A, B \in \mathfrak{U}$ . Then from

$$A \cap B = A \setminus (A \setminus B)$$

and the definition of a ring (Definition A.44) it follows that  $A \cap B \in \mathfrak{U}$ .

**Definition A.45.** A nonempty collection  $\mathfrak{U}$  of subsets of a set  $M$  is called an *algebra* if the following two conditions are satisfied:

**Condition 1** if  $A, B \in \mathfrak{U}$  then  $A \cup B \in \mathfrak{U}$ .

**Condition 2** if  $A \in \mathfrak{U}$  then its complement  $C$  equals  $M \setminus A \in \mathfrak{U}$ .

**Theorem A.77.** A necessary and sufficient condition for the collection  $\mathfrak{U}$  of subsets of a set  $M$  to be an algebra is that  $\mathfrak{U}$  be a ring and  $M \in \mathfrak{U}$ .

*Remark A.4.* The sum in Condition 1 in the Definition of an algebra A.45 may be replaced by the intersection. (It cannot be done in Definition A.44 of a ring).

**Definition A.46.** A nonempty collection  $\mathfrak{U}$  of subsets of a set  $M$  is called a  $\sigma$ -ring if  $\mathfrak{U}$  is a ring closed with respect to the union of both finite and infinite sets of sets, that is,

**Condition 3**

$$A_i \in \mathfrak{U} (i = 1, 2, \dots) \quad \text{implies that} \quad A = \bigcup_{i=1}^{\infty} A_i.$$

**Condition 4**

$$A, B \in \mathfrak{U} \quad \text{implies that} \quad A \setminus B \in \mathfrak{U}.$$

It is clear that a  $\sigma$ -ring is also closed relative to countable intersections of sets. In fact, if

$$A_i \in \mathfrak{U} \quad (i = 1, 2, \dots), \quad A = \bigcap_{i=1}^{\infty} A_i,$$

then the equation

$$A = \bigcup_{i=1}^{\infty} (A_1 \cup A_i) = A_1 \setminus \bigcup_{i=1}^{\infty} (A_1 \setminus A_i)$$

implies that  $A \in \mathfrak{U}$ .

Similarly, a notion of  $\sigma$ -algebra may be introduced. Let  $\mathfrak{U}$  be a nonempty collection of subsets of a set  $M$ . Then  $\mathfrak{U}$  is called a  $\sigma$ -algebra if Condition 3 from Definition A.46 (of a  $\sigma$ -ring) and Condition 2 from Definition A.45 (of an algebra) hold. A necessary and sufficient condition that the collection  $\mathfrak{U}$  be a  $\sigma$ -algebra is that  $\mathfrak{U}$  be a  $\sigma$ -ring and  $M \in \mathfrak{U}$ .

Suppose some set of rings (respectively algebras) consisting of the subsets of the set  $M$  is given. Then the intersection of these subsets  $\bigcap_{\alpha} \mathfrak{U}$  is also a ring. This intersection is nonempty, since  $\emptyset \in \mathfrak{U}$ . We can check, for example, that  $\mathfrak{U}$  is closed with respect to union. Let  $A, B \in \mathfrak{U}$ . Then  $A, B \in \mathfrak{U}_{\alpha}$  for any  $\alpha$ , and consequently  $A \cup B \in \mathfrak{U}_{\alpha}$  for any  $\alpha$ , and therefore  $A \cup B \in \mathfrak{U}$ . Analogously, if  $\mathfrak{U}$  is a set of  $\sigma$ -rings ( $\sigma$ -algebras), then their intersection is a  $\sigma$ -ring ( $\sigma$ -algebra), too.

If  $\mathfrak{B}$  is an arbitrary nonempty collection of subsets of the set  $M$ , then there exists the minimal ring (algebra,  $\sigma$ -ring,  $\sigma$ -algebra)  $\mathfrak{U}$ . This ring  $\mathfrak{U}$  contains  $\mathfrak{B}$  ( $\mathfrak{B} \subset \mathfrak{U}$ ). The ring  $\mathfrak{U}$  is the intersection of all rings  $\mathfrak{U}'$  (algebras,  $\sigma$ -rings, or  $\sigma$ -algebras) consisting of the subsets of the set  $M$  and containing  $\mathfrak{B}$  (an example of such a  $\mathfrak{U}'$  is the collection of all subsets of the set  $M$ ). The collection  $\mathfrak{U}$  is called the ring (algebra,  $\sigma$ -ring,  $\sigma$ -algebra) generated by the collection  $\mathfrak{B}$ .

We introduce a notion more general than that of a ring.

**Definition A.47.** A collection  $\mathfrak{R}$  of subsets of a set  $M$  is called a *semiring* if it satisfies the following conditions:

**Condition 5**

$$\emptyset \in \mathfrak{R}.$$

**Condition 6** If  $A, B \in \mathfrak{R}$ , then  $A \cap B \in \mathfrak{R}$ .

**Condition 7** If  $A, B \in \mathfrak{R}$  and  $B \subset A$ , then there exists a finite or countable collection of disjoint sets  $C_n \in \mathfrak{R}$  such that  $A \setminus B = \bigcup_n C_n$ .

An example of a semiring that is not a ring is the collection of all open, closed, and half-open intervals on the real line (including intervals consisting of one point  $[a, a]$  and the empty intervals  $(a, a)$ ). In addition, the difference of two intervals may be represented as the interval or the union of two disjoint intervals.

We single out several properties of semirings:

- If  $A, A_1, \dots, A_p \in \mathfrak{R}$  ( $\mathfrak{R}$  is a semiring), then there exists no more than one countable collection of disjoint sets  $C_n \in \mathfrak{R}$  such that

$$A \setminus \bigcup_{i=1}^p A_i = \bigcup_n C_n.$$

This assertion is a strengthened variant of Condition 7 from Definition A.47 of a semiring.

- Suppose that a set  $C$  is no more than a countable union of sets  $A_n \in \mathfrak{A}$ , i.e.,  $C = \bigcup_n A_n$ . Then  $C$  is representable as no more than a countable union of disjoint sets  $B_m \in \mathfrak{A}$ ,  $C = \bigcup_m B_m$ , and every  $B_m$  is contained in at least one  $A_n$ .

Let us illustrate the notion of semiring by Borel sets in Euclidean space.

**Definition A.48.** A set  $A \subset \mathbb{R}^n$  is called a *Borel set* if it belongs to the  $\sigma$ -algebra generated by the collection of all closed subsets from  $\mathbb{R}^n$ .

We denote the collection of all Borel sets in  $\mathbb{R}^n$  by  $\mathfrak{B}$ . Then by Definition A.48, any closed set  $F \in \mathbb{R}^n$  belongs to  $\mathfrak{B}$ ; also, every open set  $G$  belongs to  $\mathfrak{B}$ , since the open sets are the complements to the closed sets. But among the Borel sets one can meet many sets of much more complicated structure, which are neither closed nor open. For example, any set representable as a countable union of closed sets (it is said that such a set is of type  $F_\sigma$ ) is a Borel set. In particular, any countable set is representable as the union of a countable set of closed one-point sets and therefore is a Borel set. Similarly, any set representable as the intersection of a countable number of open sets (it is said that such a set is of type  $G_\delta$ ) is a Borel set, too. But the sets of type  $F_\sigma$  and type  $G_\delta$  do not exhaust the class  $\mathfrak{B}$  of all Borel sets.

### A.2.2 Additive Functions of Set

A function is called a *set function* if its domain of definition is some collection of sets. Suppose that a real function  $f$  is defined for all  $A$  belonging to a collection  $\mathfrak{A}$  of some sets, and suppose for simplicity that  $f$  is bounded from below.

A function  $f$  is called countably additive or  $\sigma$ -additive if for any finite or countable collection of disjoint sets  $A_i \in \mathfrak{A}$ ,  $\bigcup_i A_i = A \in \mathfrak{A}$ , we have the equality

$$f(A) = \sum f(A_i). \quad (\text{A.12})$$

If this equality holds only when the set  $A \in \mathfrak{A}$  is a union of finite disjoint sets  $A_i \in \mathfrak{A}$ , then the function  $f$  is called *finitely additive* or simply additive.

Let the domain of definition  $\mathfrak{A}$  of a set function  $f$  be a ring. Then the union of any finite number of sets  $A_i \in \mathfrak{A}$  belongs to  $\mathfrak{A}$ . Therefore if the equation

$$f\left(A_1 \cup A_2\right)$$

holds for any two disjoint sets  $A_1, A_2 \in \mathfrak{A}$ , then by induction, the equation (A.12) holds for any finite number of disjoint sets  $A_i \in \mathfrak{A}$ . Hence  $f$  is a finitely additive function.

Additive functions on a ring possess the following property:

**Condition 8** If  $A, B \in \mathfrak{R}$ ,  $B \subset A$ , and the value  $f(A)$  is finite, then  $f(B)$  is also finite.

Indeed, from the additivity of the function  $f$  it follows that

$$f(A) = f(B) + f(A \setminus B).$$

Suppose that  $f(B) = +\infty$ . Then  $f(A)$  is also equal to  $+\infty$ , which contradicts the hypothesis.

Let  $A, B \in \mathfrak{R}$ ,  $B \subset A$ , and suppose that  $f(B)$  is bounded. Then it follows from the previous formula that

$$f(A \setminus B) = f(A) - f(B). \quad (\text{A.13})$$

Putting  $B = A$  in (A.13), we obtain

$$f(\emptyset) = f(A) - f(A) = 0.$$

Thus the equality  $f(\emptyset) = 0$  certainly holds if there exists at least one set  $A \in \mathfrak{R}$  such that  $f(A)$  is bounded. Bounded countably additive functions may be characterized by means of the following theorems.

**Theorem A.78.** Let a bounded additive function  $f$  be defined on the ring  $\mathfrak{R}$ . Then in order that the function  $f$  be countably additive, it is necessary and sufficient that  $f(A_i) \rightarrow 0$  for any decreasing sequence of sets  $A_i \in \mathfrak{R}$  ( $i = 1, 2, \dots$ ) with empty intersection (i.e.,  $\bigcap_{i=1}^{\infty} A_i = \emptyset$ ).

**Theorem A.79.** Let  $f$  be a countably additive function on the ring  $\mathfrak{R}$ . If  $A \in \mathfrak{R}$ ,  $A_i \in \mathfrak{R} \cup_{i=1}^{\infty} A_i = A$ , and the sets  $A_i$  form the increasing sequence  $A_1, A_2, \dots$ , then the relation

$$f(A) = \lim_{i \rightarrow \infty} f(A_i) \quad (\text{A.14})$$

must hold.

The equality (A.14) is also true if  $A = \bigcap_{i=1}^{\infty} A_i$ ,  $(A, A_i) \in \mathfrak{R}$  and the  $A_i$  form a decreasing sequence, and  $f(A_i)$  is bounded beginning with some  $i$ .

### A.2.3 Measure and Its Properties

**Definition A.49.** Let  $X$  be an arbitrary set. A *measure* on  $X$  is a real nonnegative additive function  $m$  defined on a semiring  $\mathfrak{R}$  of subsets of the set  $X$  such that  $m(\emptyset) = 0$ .

A measure  $m$  is called *finite* if  $m(A) < +\infty$  for all  $A \in \mathfrak{R}$ . A measure  $m$  is called  *$\sigma$ -finite* if  $\forall A \in \mathfrak{R}$  there exist  $A_n \in \mathfrak{R}$ ,  $n = 1, 2, \dots$ , such that  $A \in \bigcup_n A_n$  and  $m(A_n) < +\infty \forall n$ .

Note that for  $\sigma$ -finite measures the condition  $m(\emptyset) = 0$  may be obtained from the other properties of measure. But this is not so in the general case: a nonnegative countably additive function may be equal to  $\infty$  identically. Consider some properties of measures that follow from its definition:

- (1) If we are given an at most countable family of disjoint sets  $A_n \in \mathfrak{R}$  ( $n = 1, 2, \dots$ ) such that  $A_n \subset A \in \mathfrak{R}$  for any  $n$ , then

$$\sum_n m(A_n) \leq m(A).$$

A special case of property (1) is the monotonicity of measure: if  $A, B \in \mathfrak{R}$  and  $B \subset A$ , then  $m(B) \leq m(A)$ .

- (2) Let  $A, A_n \in \mathfrak{R}$ ,  $n = 1, 2, \dots$ ,  $A \subset \bigcup_n A_n$ , and suppose the collection of sets  $A_n$  is finite or countable. Then

$$m(A) \leq \sum_n m(A_n).$$

Property (2) is called *countable semiadditivity* of measure.

### A.2.4 Outer Measure

**Definition A.50.** Let  $X$  be an arbitrary set. A real nonnegative function  $\mu^*$  on the collection of all subsets of the set  $X$  is called an *outer measure* on  $X$  if it satisfies the following conditions:

**Condition 9**

$$\mu^*(\emptyset) = 0.$$

**Condition 10** If  $E \subset \bigcup_n E_n$ ,  $E_n \subset X$ , then  $\mu^*(E) \leq \sum_n \mu^*(E_n)$ .

Condition 10 is called *countable semiadditivity* of the outer measure. In particular, countable semiadditivity of the outer measure implies its monotonicity: if  $E \subset E_1$ , then  $\mu^*(E) \leq \mu^*(E_1)$ .

Note that the outer measure does not need to be an additive or even finite function of set. And note that the measure on  $X$  is an outer measure if it is defined on the collection of all subsets of  $X$ .

The next two constructions give the method of an outer measure construction if a measure in  $X$  is defined, and conversely, a measure construction method if an outer measure is defined.

**Theorem A.80.** Let  $m$  be a measure on  $X$  defined on a semiring  $\mathfrak{R}$ , and let  $\mu^*$  be the set function defined for any  $E \subset X$  according to the following rule:

(1) If there exists an at most countable covering of  $E$  by sets belonging to the semiring  $\mathfrak{A}$ , i.e.,  $E \subset \bigcup_n A_n$ , where  $A_n \in \mathfrak{A}$  ( $n = 1, 2, \dots$ ), then

$$\mu^*(E) = \inf\left\{\sum_n m(A_n)\right\},$$

where the lower bound is taken over all such coverings;

(2) otherwise,  $\mu^*(E) = +\infty$ .

Then  $\mu^*$  is an outer measure in  $X$  and also  $\mu^*(A) = m(A) \forall A \in \mathfrak{A}$ .

In what follows we say that the outer measure  $\mu^*$  constructed according to Theorem A.80 is generated by the measure  $m$ .

We introduce the following notation: if  $E$  is an arbitrary subset of  $X$  then  $E' = X \setminus E$ , i.e.,  $E'$  is the complement of  $E$  in  $X$ .

Now let  $\mu^*$  be an arbitrary outer measure in  $X$ . Take two sets  $A, B \subset X$ . We shall agree to say that the set  $A$  "divides well" the set  $E$  if

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A'). \quad (\text{A.15})$$

Since the outer measure is countably semiadditive, the sign  $\leq$  always occurs in (A.15). Therefore, to say that the set  $A$  divides well the set  $B$ , it is enough to establish the validity of the inequality

$$\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A'). \quad (\text{A.16})$$

But (A.16) needs to be verified only in case  $\mu^*(E) < +\infty$ . If the set  $A$  divides well the set  $E$ , then  $A'$  also divides well  $E$ , since (A.15) is symmetric with respect to  $A$  and  $A'$ . We call the set  $A \subset X$   $\mu^*$ -measurable if  $A$  divides well any set  $E \subset X$ . Denote by  $\mu$  a contraction of the outer measure on a collection of all  $\mu^*$ -measurable sets.

**Theorem A.81.** *The collection  $\mathfrak{G}$  of all  $\mu^*$ -measurable sets in  $X$  is a  $\sigma$ -algebra, and the function  $\mu$  is a measure in  $X$ .*

We say further that the measure  $\mu$  is generated by the outer measure  $\mu^*$ . We single out some properties of  $\mu^*$ -measurable sets.

- (1) If  $\mu^*(A) = 0$  then  $A \in \mathfrak{G}$ .
- (2) If  $A \in \mathfrak{G}$ ,  $\mu(A) = 0$ , and  $E \subset A$ , then  $E \in \mathfrak{G}$ .
- (3) If  $A_1 \subset A \subset A_2$ , and also  $A_1, A_2 \in \mathfrak{G}$ ,  $\mu(A_1) = \mu(A_2) < +\infty$ , then  $A \in \mathfrak{G}$  (and moreover,  $\mu(A) = \mu(A_1) = \mu(A_2)$  by the monotonicity of measure).
- (4) *Criterion for  $\mu^*$ -measurability.* Let  $E \subset X$ . If for any  $\varepsilon > 0$  there exist  $A, B \in \mathfrak{G}$  such that  $A \subset E \subset B$  and  $\mu(B \setminus A) < \varepsilon$ , then  $E \in \mathfrak{G}$ .

### A.2.5 Standard Extension of Measure

**Theorem A.82.** *Let  $m$  be a measure given on a semiring  $\mathfrak{R} \subset X$ ,  $\mu^*$  the outer measure generated by the measure  $m$ ,  $\mu$  the measure generated by the measure  $\mu^*$ . Then  $\mu$  is the extension of the measure  $m$  to  $\mathfrak{G}$  of  $\mu^*$ -measurable sets, i.e.,  $\mathfrak{R} \subset \mathfrak{G}$  and  $m(A) = \mu(A)$  if  $A \in \mathfrak{R}$ .*

In the sequel, we shall call the measure  $\mu$  obtained in such a way the standard extension of the measure  $m$  (or Carathéodory extension); we shall also call  $\mu^*$ -measurable sets  $m$ -measurable.

*Remark A.5.* If  $X = \bigcup_{n=1}^{\infty} A_n$ ,  $A_n \in \mathfrak{R}$ , for all  $n$  and  $m(A_n) < +\infty$ , then the measure  $m$  and its standard extension  $\mu$  are  $\sigma$ -finite. Obviously the inverse is true as well: if  $\mu$  is  $\sigma$ -finite, then  $m$  is  $\sigma$ -finite, and the set  $X$  admits the above-mentioned representation.

The simplest properties of  $m$ -measurable sets have already been mentioned. In particular, any subset  $E$  of an  $m$ -measurable set  $A$  with  $\mu(A) = 0$  is  $m$ -measurable (and  $\mu(E) = 0$ ). In this connection we introduce the following general notion.

**Definition A.51.** Let  $m$  be an arbitrary measure in  $X$  given on a semiring  $\mathfrak{R} \subset X$ . This measure is called a *complete measure* if  $A \in \mathfrak{R}$ ,  $m(A) = 0$ , and  $E \subset A$  imply that  $E \subset \mathfrak{R}$ .

We note one general property of a complete measure.

**Theorem A.83.** *Let the domain of definition  $\mathfrak{R}$  of the complete measure  $m$  in  $X$  be a  $\sigma$ -algebra. If  $E \subset X$ , and  $\forall \varepsilon > 0$ ,  $\exists A \in \mathfrak{R}$  such that  $E \subset A$  and  $m(A) < \varepsilon$ , then  $E \in \mathfrak{R}$  and  $m(E) = 0$ .*

Note that the standard extension may not be minimal in the sense that the  $\sigma$ -algebra  $\mathfrak{G}$  of  $m$ -measurable sets may be wider than the  $\sigma$ -algebra generated by the semiring  $\mathfrak{R}$ .

It turns out that the Carathéodory reiteration procedure gives nothing new, as the next theorem tells us.

**Theorem A.84.** *Let  $m$  be a measure in  $X$  given on a semiring  $\mathfrak{R}$ ,  $\mu$  its standard extension, and  $\mu^*$  and  $\gamma^*$  outer measures generated by the measures  $m$  and  $\mu$  respectively. Then*

$$\gamma^*(E) = \mu^*(E), \quad \forall E \subset X.$$

From this it follows that the collection of all  $\mu$ -measurable sets coincides with the collection of all  $m$ -measurable sets, and therefore, the standard extension of measure  $\mu$  coincides with the measure  $\mu$  itself.

### A.2.6 Uniqueness of Extension of Measure

If some measure  $m$  is given on a semiring  $\mathfrak{R} \subset X$ , it is possible that besides its standard extension  $\mu$  to the  $\sigma$ -algebra  $\mathfrak{G}$  of  $m$ -measurable sets, there exist some other extensions of this measure to some other  $\sigma$ -algebras. However, under some restrictions on the limits of the  $\sigma$ -algebra  $\mathfrak{G}$ , there is a unique extension. A precise formulation of this result will be given after the next lemma, which is itself of interest.

**Lemma A.1.** *Let  $E \subset X$  and  $\mu^*(E) < +\infty$ . Then there exists a decreasing sequence of sets  $H_n \in \mathfrak{G}$ ,  $n = 1, 2, \dots$ , satisfying the following conditions:*

**Condition 11**

$$E \subset H_n, \quad \mu(H_n) < +\infty, \quad \forall n;$$

**Condition 12**  $\forall n$ ,  $H_n$  is representable in the form of an at most countable union of disjoint sets  $T_{nk}$  from the semiring  $\mathfrak{R}$ , i.e.,  $H_n = \bigcup_k T_{nk}$ .

**Condition 13** If  $H = \bigcap_{n=1}^{\infty} H_n$  then  $\mu(H) = \mu^*(E)$ .

It follows immediately from Condition 12 that the sets  $H_n$  and together with them the set  $H$  are contained in the  $\sigma$ -algebra generated by the semiring  $\mathfrak{R}$ .

**Theorem A.85.** *Let  $\mu$  be a standard extension of the measure  $m$  from the semiring  $\mathfrak{R}$  to the  $\sigma$ -algebra  $\mathfrak{G}$ . Suppose the measure  $\mu$  is  $\sigma$ -finite and that the measure  $\gamma$  is the extension of the measure  $m$  to some  $\sigma$ -algebra  $\mathfrak{T} \subset \mathfrak{R}$ . Then  $\mu(A) = \gamma(A)$  for all  $A \in \mathfrak{G} \cap \mathfrak{T}$ . If in addition to the foregoing conditions,  $\gamma$  is a complete measure, then  $\mathfrak{G} \subset \mathfrak{T}$ .*

There is one more characteristic of the standard extension  $\mu$  of the measure  $m$  from a semiring  $\mathfrak{R}$ . This property holds only when  $X$  can be covered by a countable collection of sets  $A_n \in \mathfrak{R}$ ,  $\mu(A_n) < +\infty$ . Then, according to Theorem A.85,  $\mu$  is the complete measure in  $X$  that is an extension of the measure  $m$  from the semiring  $\mathfrak{R}$  to the smallest possible  $\sigma$ -algebra  $\mathfrak{T} \subset \mathfrak{R}$  (as we know, in this case  $\mathfrak{T} = \mathfrak{G}$ ).

### A.2.7 The Lebesgue Measure in Euclidean Space

Let  $n$  pairs of real numbers  $a_i$  and  $b_i$ ,  $i = 1, 2, \dots, n$ , be given in Euclidean space such that  $a_i < b_i$  for all  $i$  (some numbers may be singular, i.e.,  $a_i = -\infty$  or  $b_i = +\infty$  for some  $i$ ). Then the collection  $\Delta_0$  of all points  $x = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}_n$  that satisfy the inequalities  $a_i < \xi_i < b_i$ ,  $i = 1, 2, \dots, n$ , is called an open ( $n$ -dimensional) parallelepiped. The collection  $\Delta^*$  of all points  $\chi \in \mathbb{R}_n$  that satisfy

the inequality  $a_i \leq \xi_i \leq b_i$ ,  $i = 1, 2, \dots, n$ , is called a closed ( $n$ -dimensional) parallelepiped.

It follows from the chosen metrics in  $\mathbb{R}_n$  that every closed (open) parallelepiped is a closed (open) set. It is clear that  $\Delta_0$  is the set of all inner points from  $\Delta^*$ , and  $\Delta^* = \bar{\Delta}_0$ , i.e.,  $\Delta^*$  is the closure of  $\Delta_0$ . If we consider as boundary points of an open set all its limit points not contained in it, then we say that  $\Delta^*$  is obtained from  $\Delta_0$  by adding to the latter all its boundary points.

Any set  $\Delta$  obtained from  $\Delta_0$  by association of some part of its boundary points is also called an  $n$ -dimensional parallelepiped. We say that  $\Delta$  is a parallelepiped with finite edges if  $-\infty < a_i < b_i < +\infty \forall i$ , and a parallelepiped with infinite edges otherwise.

Consider an arbitrary parallelepiped  $\Delta = \langle a_1, b_1; a_2, b_2; \dots; a_n, b_n \rangle$ , and write  $\Delta = (a_1, b_1; a_2, b_2; \dots; a_n, b_n)$  if the parallelepiped is open and

$$\Delta = [a_1, b_1; a_2, b_2; \dots; a_n, b_n]$$

if  $\Delta$  is closed, or for short,  $\Delta = \langle a, b \rangle$ ,  $\Delta = (a, b)$ ,  $\Delta = [a, b]$  respectively.

Suppose that each interval  $\langle a_i, b_i \rangle$  is divided into a finite number of intervals

$$a_i = a_i^{(0)} < a_i^{(1)} < \dots < a_i^{(k_i)} = b_i. \tag{A.17}$$

We form the parallelepipeds

$$\Delta_{j_1, j_2, \dots, j_n} = \langle a_1^{(j_1)}, a_1^{(j_1+1)}; \dots; a_n^{(j_n)}, a_n^{(j_n+1)} \rangle, \tag{A.18}$$

where  $j_i = \overline{(0, k_i - 1)}$ . In all, there are  $k_1, k_2, \dots, k_n$  such parallelepipeds.

We say that parallelepipeds (A.18) in aggregate form a *network partition* of the parallelepiped  $\Delta$ .

**Definition A.52.** The product

$$v\Delta = \prod_{i=1}^n (b_i - a_i) \tag{A.19}$$

is called the *volume* of the parallelepiped  $\Delta = \langle a, b \rangle$ .

**Lemma A.2.** *If a parallelepiped  $\Delta$  is formed as a union of a finite number of disjoint parallelepipeds  $\Delta_k$ ,  $\Delta = \bigcup_{k=1}^p \Delta_k$ , then*

$$v\Delta = \sum_{k=1}^p v\Delta_k. \tag{A.20}$$

**Lemma A.3.** *If disjoint parallelepipeds  $\Delta_k$ ,  $k = 1, 2, \dots, p$ , are contained in the parallelepiped  $\Delta$ ,  $\bigcup_{k=1}^p \Delta_k \in \Delta$ , then*

$$\sum_{k=1}^p vA_k \leq v\Delta. \quad (\text{A.21})$$

**Lemma A.4.** *If there are such not necessarily disjoint parallelepipeds  $\Delta_k$ ,  $k = 1, 2, \dots, p$ , such that  $\Delta \subset \bigcup_{k=1}^p \Delta_k$ , then*

$$v\Delta \leq \sum_{k=1}^p v\Delta_k. \quad (\text{A.22})$$

**Lemma A.5.** *Suppose that the parallelepiped  $\Delta$  is given as the countable union of pairwise disjoint parallelepipeds  $\Delta_k$ , i.e.,  $\Delta = \bigcup_{k=1}^{\infty} \Delta_k$ . Then*

$$v\Delta = \sum_{k=1}^{\infty} v\Delta_k. \quad (\text{A.23})$$

From Lemmas A.2 and A.5 it follows that  $v$  is a countably additive function of  $\Delta$ . But parallelepipeds do not form a semiring, and therefore the function  $v$  is not a measure.

**Definition A.53.** A parallelepiped  $\Delta = \langle a, b \rangle$  is called an  $n$ -dimensional *cell* if it consists of all points whose coordinates satisfy the inequalities  $a_i \leq \xi_i \leq b_i$ ,  $i = 1, 2, \dots, n$ .

Denote a cell by  $\Delta = [a_1, b_1, a_2, b_2, \dots, a_n, b_n]$  or briefly  $\Delta = [a, b]$ .

The collection of all cells in the space  $\mathbb{R}_n$  is a semiring. In the sequel, this semiring will be denoted by the symbol  $\mathfrak{R}$ .

**Theorem A.86.** *A function  $v$  defined on the semiring  $\mathfrak{R}$  that for every cell is equal to the volume of that cell is a  $\sigma$ -finite measure on  $\mathbb{R}_n$ .*

**Theorem A.87.** *Any open set  $G \subset \mathbb{R}_n$  can be represented as an at most countable union of disjoint  $n$ -dimensional cells with finite edges.*

### A.2.8 Measurable Sets

**Definition A.54.** The standard extension  $\mu$  of the volume  $v$  is called *Lebesgue measure* (in the sequel we call it simply the measure on  $\mathbb{R}_n$ ). The sets for which the measure  $\mu$  is defined (i.e., the  $v$ -measurable sets) are called *Lebesgue measurable* (or simply measurable).

Other measures may be defined on  $\mathbb{R}_n$ , but in what follows if there is no special remark we mean Lebesgue measure.

The measure  $\mu$  is  $\sigma$ -finite, since  $\mathbb{R}_n$  may be presented in the form of a countable union of cells with finite volumes. The outer measure  $\mu^*$  generated by the

measure  $\nu$  is defined on all sets from  $\mathbb{R}_n$ . The measure  $\mu^*$  is called the Lebesgue outer measure. The measure  $\mu$  is, by construction, the contraction of the outer measure  $\mu^*$  to the  $\sigma$ -algebra  $\mathfrak{G}$  of measurable sets.

The collection of all measurable sets forms a  $\sigma$ -algebra, and therefore the union and the intersection of a finite or countable number of measurable sets are measurable and the difference of two measurable sets is measurable; in particular, the complement of a measurable set with respect to the whole of  $\mathbb{R}_n$  is measurable.

It follows from the general properties of measure that Lebesgue measure is countably semiadditive and countably additive. As a standard extension of the volume  $\nu$ , the Lebesgue measure is complete, i.e., any subset of a set of measure zero is measurable (and is also of measure zero).

Since Lebesgue measure is generated by the outer measure  $\mu$ , the general criterion for  $\mu$ -measurability (see (A.2.4)) applies to it. In what follows we shall simply refer to this criterion as a measurability criterion in  $\mathbb{R}_n$ .

By construction, the  $\sigma$ -algebra of measurable sets includes all the cells, and  $\mu(\Delta) = V(\Delta)$  for any cell  $\Delta$ .

**Theorem A.88.** *Every open set and every closed set in  $\mathbb{R}_n$  is measurable.*

**Theorem A.89.** *Any parallelepiped  $\Delta$  is measurable and  $\mu(\Delta) = \nu(\Delta)$ .*

**Theorem A.90.** *Every finite or countable set of points in  $\mathbb{R}_n$  is measurable, its measure being equal to zero.*

**Theorem A.91.** *All Borel sets in  $\mathbb{R}_n$  are measurable.*

It should be noted that the  $\sigma$ -algebra  $\mathfrak{G}$  is not exhausted by Borel sets. Recall that the standard extension may not be minimal. In  $\mathbb{R}_n$  a standard extension of volume  $\nu$  from the semiring of cells would be its extension to the  $\sigma$ -algebra  $\mathfrak{B}$  of the Borel sets, that is, contraction of the Lebesgue measure to  $\mathfrak{B}$ .

**Theorem A.92.** *The outer measure of any set  $E \subset \mathfrak{R}_n$  is equal to the exact lower boundary of the measures of all the possible open sets  $G$  containing  $E$ :*

$$\mu^*(E) = \inf_{E \subset G} \mu(G). \quad (\text{A.24})$$

**Corollary A.3.** *For any measurable set  $E \subset G$  and any  $\varepsilon > 0$  there exist*

- *an open set  $G \subset \mathfrak{R}_n$  such that  $E \subset G$  and  $\mu(G \setminus E) < \varepsilon$ ;*
- *a closed set  $F \subset \mathfrak{R}_n$  such that  $F \subset E$  and  $\mu(E \setminus F) < \varepsilon$ .*

**Corollary A.4.** *The measure of any measurable set  $E \subset \mathfrak{R}_n$  is equal to the exact upper bound of the measures of all the possible closed sets  $F$  contained in  $E$ :*

$$\mu(E) = \sup_{F \subset E} \mu(F). \quad (\text{A.25})$$

**Theorem A.93.** For any measurable set  $E \subset \mathcal{R}_n$  there exist an  $F_\sigma$ -set  $H$  and a  $G_\sigma$ -set  $K$  (see Section A.2.1) such that

$$H \subset E \subset K, \quad \mu(H) = \mu(E) = \mu(K), \quad \mu(K \setminus H) = 0.$$

**Corollary A.5.** Every measurable set is representable as a union of Borel  $F_\sigma$ -sets and some measurable set of measure zero.

A mapping of the space  $\mathcal{R}_n$  onto itself is said to be *isometric* (length-preserving) if the distance between any two points is equal to the distance between the images of these points under the mapping. Sets  $E_1$  and  $E_2$  belonging to  $\mathcal{R}_n$  are called congruent if one of them is the image of the other under an isometric mapping. It can be proved that if two sets  $E_1$  and  $E_2$  are congruent and one of them is measurable, then the other is also measurable and  $\mu(E_1) = \mu(E_2)$ . In other words, the measure is invariant with respect to isometric mappings.

It is known that in each space  $\mathcal{R}_n$  there exist sets that are not Lebesgue measurable. Moreover, it has been proved that in any space  $\mathcal{R}_n$  it is impossible to construct a  $\sigma$ -finite measure  $m$  such that

1.  $m$  is defined for all the sets from  $\mathcal{R}_n$ ;
2.  $m$  is invariant with respect to the length-preserving mapping;
3. the measure  $m$  of any parallelepiped coincides with its volume.

The condition of  $\sigma$ -additivity that we included in the definition of measure plays an essential role in the above proposition. If we generalize slightly the concept of a measure, admitting that the measure should only be finitely additive, then as was proved by Banach, in spaces  $\mathcal{R}_1$  and  $\mathcal{R}_2$  there exist measures possessing properties 1–3. But in case  $n \geq 3$ , even a finitely additive measure in  $\mathcal{R}_n$  satisfying conditions 1–3 does not exist. The latter result was established by Hausdorff.

### A.2.9 Measurable Functions

We shall begin with real functions with finite values whose domain of definition can be an arbitrary subset of  $X$ . As a rule, in what follows this subset will be supposed measurable.

**Definition A.55.** Let a function  $f$  be defined on a set  $E \subset X$ . The sets of the four types

1.  $E[f(x) > a]$ ,
2.  $E[f(x) \geq a]$ ,
3.  $E[f(x) < a]$ ,
4.  $E[f(x) \leq a]$ ,

where  $a$  can be any real number, are called *Lebesgue sets* of the function  $f$ .

**Definition A.56.** A function  $f$  defined on a set  $S \subset \mathbf{X}$  is called *measurable* (on this set) or, more precisely,  $\mathfrak{G}$ -*measurable*, if all the Lebesgue sets of the function  $f$  belonging to the four enumerated types in Definition A.55 are measurable for any  $a$ .

If  $\mathbf{X} = \mathcal{R}_n$  and  $\sigma$  consists of the Lebesgue measurable sets, then a measurable function is called *Lebesgue measurable*.

Note that if a function  $f$  is measurable on  $E$ , the set  $E$  is measurable. If  $f$  is measurable on  $E$ , the set  $E[f(x) = a]$  is also measurable for every  $a$ .

**Theorem A.94.** *If a function  $f$  defined on a set  $E$  is such that its Lebesgue sets of one of the four given types are measurable for all  $a$ , then this function is measurable.*

Let us formulate some simple propositions.

**Proposition A.1.** (1) *If  $f(x) = c$  (where  $c$  is constant) on a measurable set  $E$ , the function  $f$  is measurable.*

(2) *If  $f$  is measurable on a set  $E$ , it is also measurable on every measurable subset  $E' \subset E$ .*

(3) *Let a function  $f$  be defined on a set  $E$  representable as a finite or countable union of sets  $E_i$  ( $E = \bigcup_i E_i$ ). If  $f$  is measurable on each  $E_i$ , it is also measurable on  $E$ .*

A special case of the last proposition reads as follows: Let  $E = \bigcup_i E_i$ , where all  $E_i$ ,  $i = 1, 2, \dots$ , are measurable. If a function  $f$  defined on  $E$  assumes a constant value on each  $E_i$ , then  $f$  is measurable.

**Definition A.57.** Let  $E \subset X$ . The function  $\chi_E$  determined by the rule

$$\chi_E(x) = \begin{cases} 1 & \text{for } x \in E, \\ 0 & \text{for } x \in X \setminus E, \end{cases}$$

is called the *characteristic function* of the set  $E$ .

**Proposition A.2.** *If  $E$  is a measurable set its characteristic function  $\chi_E$  is measurable (on  $X$ ).*

It is easy to prove that the converse is also true.

**Proposition A.3.** *If  $\mu$  is a complete measure, then any function  $f$  defined on a set  $E$  with  $\mu(E) = 0$  is measurable.*

**Theorem A.95.** *If a function  $f$  is continuous on a closed set  $F \subset \mathcal{R}_n$ , it is Lebesgue measurable on that set.*

**Corollary A.6.** *If  $f$  is a continuous function on a measurable set  $E \subset \mathcal{R}_n$ , it is measurable on  $E$ .*

If the space  $\mathcal{R}_n$  is again taken as the set  $X$  while the  $\sigma$ -algebra  $\mathfrak{B}$  of all Borel sets in  $\mathcal{R}_n$  is taken as  $\mathfrak{G}$ , Definition A.56 leads to the notion of  $\mathfrak{B}$ -measurable functions. The  $\sigma$ -measurable functions in  $\mathcal{R}_n$  are also called *Baire functions*.

Theorem A.91 shows that all Baire functions are Lebesgue measurable. It is easy to see that a function continuous on a closed set from  $\mathcal{R}_n$  is  $\mathfrak{B}$ -measurable on this set.

**Theorem A.96.** *If  $f$  and  $g$  are two measurable functions on a set  $E \in X$ , then functions  $kf$  ( $k$  any constant),  $|f|$ ,  $f^2$ ,  $f \pm g$ , and  $fg$  are measurable on  $E$ . If, moreover,  $f(x) \neq 0$  on  $E$ , then the function  $1/f$  is measurable on  $E$  as well, and when  $g(x) \neq 0$ , then the function  $f/g$  is also measurable on  $E$ .*

**Corollary A.7.** *If  $f$  and  $g$  are measurable on  $E$ , then both sets  $A = E[f(x) = g(x)]$  and  $B = \bar{E}[f(x) \neq g(x)]$  are measurable.*

**Theorem A.97.** *If functions  $f_k$ ,  $k = 1, 2, \dots$ , and  $f$  are defined on a set  $E \in X$ , the functions  $f_k$  being measurable on  $E$  for all  $k$ , and  $f(x) = \lim f_k(x)$  for each  $x \in E$ , then the function  $f$  is also measurable on the set  $E$ .*

**Corollary A.8.** *If functions  $u_k$ ,  $k = 1, 2, \dots$ , are measurable functions on a set  $E$ , if  $s$  is a function defined on a set  $E$ , and if the limiting relation*

$$s(x) = \sum_{k=1}^{\infty} u_k(x)$$

*holds for all  $x \in E$ , then the function  $s$  is also measurable on  $E$ .*

Let us consider functions defined on a closed set  $F \in \mathcal{R}_n$ . As is already known (Theorem A.95), all continuous functions are measurable on  $F$ . It is also known that if a sequence of continuous functions is uniformly convergent, then the limit function is also continuous. But if a sequence of continuous functions converges at each point but not uniformly, the limit function is no longer necessarily continuous. According to the classification of functions introduced by Baire, a discontinuous function serving as a limit of a sequence of continuous functions is said to be of the *first Baire class*. By Theorem A.97, functions of the first class are measurable (and even  $\mathfrak{B}$ -measurable).

This classification of discontinuous functions can be continued as follows. A function not belonging to Baire class 1 and representable as a pointwise limit of a sequence of functions of class 1 is of Baire class 2. A simple example of a function of Baire class 2 in the space  $\mathcal{R}_1$  is the well-known Dirichlet function

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

This function can be obtained from continuous functions by means of the twofold limit

$$f(x) = \lim_{m \rightarrow \infty} \lim_{k \rightarrow \infty} \cos^{2k}(m! \pi x).$$

Similarly, proceeding from functions of class 2, we can define functions of Baire class 3, etc. All the functions obtained with the aid of such subsequent passages to the limit turn out to be  $\mathfrak{B}$ -measurable (i.e., they are Baire functions).

But the collection of all functions obtained from continuous functions by means of an arbitrary finite number of passages to the limit does not exhaust the class of all Baire functions. The latter class, however, is essentially narrower than the class of all Lebesgue measurable functions.

It is sometimes useful to assume that functions can take the values  $+\infty$  and  $-\infty$ . The repetition of Definitions A.55 and A.56 leads to a definition of measurability for such functions as well. Note that there is no need to assume that  $a$  can be equal to  $+\infty$  or  $-\infty$ : if  $f$  is measurable on  $E$ , both sets  $\{x \in E : f(x) = +\infty\}$  and  $\{x \in E : f(x) = -\infty\}$  are measurable.

It can be easily verified that the theorems and propositions established in this section hold for functions assuming infinite values as well, with the only caveat that when we speak of operations on functions it is required that these operations make sense. The only essential exception to this rule is Theorem A.94, the statement of which should additionally involve the condition that the set  $E$  be measurable. Although, as before, the measurability of a function taking infinite values implies the measurability of the set  $E$  in this theorem, the measurability of the set  $E$  (in contrast to the case in which the function takes only finite values) does not follow from the assumption that only the Lebesgue sets of one certain type are measurable.

Let us agree that in what follows, the functions in question can take both finite and infinite values.

**Theorem A.98.** *If functions  $f_k$ ,  $k = 1, 2, \dots$ , and  $f$  are measurable on a set  $E \subset X$ , the subset  $E[f_k(x) \rightarrow f(x)]$  is also measurable.*

### A.2.10 Equivalent Functions, Convergence Almost Everywhere, and Convergence in Measure

**Definition A.58.** Let a certain property hold at some points of a set  $E \subset X$  and not hold at other points. If the points of  $E$  for which the property does not hold are contained in a subset of measure zero, this property is said to hold *almost everywhere* on the set  $E$  (or for almost all  $x$  belonging to  $E$ ).

*Remark A.6.* If we deal with a finite or countable number of some properties each of which holds almost everywhere on a set  $E$ , all these properties also hold simultaneously almost everywhere on  $E$ .

If the measure  $\mu$  defined on  $X$  is complete, the above definition is simplified. Namely, in this case, “almost everywhere on  $E$ ” means “for all  $x \in E$  except possibly on a subset  $E' \subset E$  with  $\mu(E') = 0$ .”

In the general theory of integration it is often allowable not to take into account some peculiarities of the function in question that pertain only to a set of measure zero.

**Definition A.59.** Two functions  $f$  and  $g$  defined on a set  $E \subset X$  are said to be *equivalent* (written  $f \sim g$ ) on the set  $E$  if  $f(x) = g(x)$  almost everywhere on  $E$ .

**Theorem A.99.** Let the measure  $\mu$  on  $X$  be complete. Then if two functions  $f$  and  $g$  are equivalent on a set  $E \subset X$  and one of them is measurable on that set, then the other is also measurable.

**Theorem A.100.** Let the measure  $\mu$  in  $X$  be complete. If  $f_k$ ,  $k = 1, 2, \dots$ , and  $f$  are defined on a set  $E \subset X$ , the  $f_k$  are measurable on  $E$  for all  $k$ , and  $f_k \rightarrow f(x)$  almost everywhere on  $E$ , then the function  $f$  is also measurable on  $E$ .

*Remark A.7.* Both these theorems fail to hold if the condition of completeness of the measure is omitted. But in this case the following assertion holds: if  $f_k$ ,  $k = 1, 2, \dots$ , are measurable functions on  $E$  and  $f_k(x) \rightarrow f(x)$  almost everywhere on  $E$ , then there exists a measurable function  $g$  defined on  $E$  such that  $f_k(x) \rightarrow g(x)$  almost everywhere on  $E$  and  $g \sim f$ . Without the requirement that the measure be complete, the following property holds: if  $f_k$  and  $f$  are measurable on  $E$  and  $f_k(x) \rightarrow f(x)$  almost everywhere on  $E$ , then the set

$$E[f_k(x) \not\rightarrow f(x)]$$

is measurable and is of measure zero. (This follows from Theorem A.98.)

**Definition A.60.** Let functions  $f_k$ ,  $k = 1, 2, \dots$ , and  $f$  be measurable on a set  $E \subset X$ . The sequence  $\{f_k\}$  is said to *converge in measure* to the function  $f$  (on the set  $E$ ) if

$$\mu(E[|f_k(x) - f(x)| \geq \varepsilon]) \xrightarrow{k \rightarrow \infty} 0$$

for every  $\varepsilon > 0$ .

For convergence in measure we shall use the notation

$$f_k \implies f.$$

**Theorem A.101.** If  $f_k \implies f$  and  $f_k \implies g$  on  $E$ , then  $f \sim g$  on  $E$ .

**Theorem A.102 (Lebesgue's theorem).** If a sequence of functions  $f_k$ ,  $k = 1, 2, \dots$ , measurable on a set  $E$ , with  $\mu(E) < +\infty$ , is convergent almost everywhere on that set to a measurable function  $f$ , then  $f_k \implies f$ .

*Remark A.8.* The converse to Lebesgue's theorem is false: a sequence can be convergent in measure while at the same time having no limit at every point. The condition  $\mu(E) < +\infty$  is also essential and cannot be omitted, that is, the convergence of a sequence of functions almost everywhere on a set  $E$  with  $\mu(E) = +\infty$  does not, in general, imply convergence in measure.

**Theorem A.103.** *If  $f_k \implies f$  on a set  $E$  and simultaneously,  $f_k(x) \rightarrow g(x)$  almost everywhere on  $E$ , then  $f \sim g$  on  $E$ .*

**Theorem A.104 (F. Riesz).** *If  $f_k \implies f$  on a set  $E$ , then there exists a subsequence  $\{f_{k_i}\}$  convergent to  $f$  almost everywhere on  $E$ .*

**Theorem A.105 (Stability of convergence).** *If  $f_k$ ,  $k = 1, 2, \dots$ , are measurable functions on a set  $E \subset X$  with  $\mu(E) < +\infty$  and are finite almost everywhere on  $E$ , and if  $f_k(x) \rightarrow 0$  almost everywhere on  $E$ , there is an increasing sequence of positive numbers  $\lambda_k \rightarrow +\infty$  such that  $\lambda_k f_k(x) \rightarrow 0$  almost everywhere on  $E$ .*

**Theorem A.106 (Convergence regulator).** *If  $f_k$ ,  $k = 1, 2, \dots$ , are measurable and finite almost everywhere on a set  $E \subset X$  with  $\mu(E) < +\infty$  and if  $f_k(x) \rightarrow \infty$  almost everywhere on  $E$ , then there exist a measurable nonnegative function  $g$  finite almost everywhere on  $E$  and a sequence of positive numbers  $\alpha_k \rightarrow 0$  such that*

$$|f_k(x)| \leq \alpha_k g(x)$$

for almost all  $x \in E$ .

This theorem shows that convergence almost everywhere turns out to be "uniform" with respect to some function  $g$ . Such a function is called a *convergence regulator* of the sequence  $\{f - k\}$ .

**Theorem A.107 (D. F. Egorov).** *Let  $\{f_k\}$  be a sequence of measurable functions finite almost everywhere on a set  $E \subset X$  (with  $\mu(E) < +\infty$ ) converging to a function  $f$  that is also finite almost everywhere. Then for any  $\varepsilon > 0$  there exists a measurable set  $E' \subset E$  such that  $\mu(E') < \varepsilon$  and  $\{f_k\}$  converges to  $f$  uniformly on the set  $E \setminus E'$ .*

This theorem shows that for measurable functions finite almost everywhere, convergence almost everywhere "does not essentially differ" from uniform convergence.

All the next theorems will be established for the special case of functions defined in the Euclidean space  $\mathbb{R}_n$  of an arbitrary dimension  $n$ , and the letter  $\mu$  will denote the Lebesgue measure on  $\mathbb{R}_n$ .

**Theorem A.108 (N. N. Luzin).** *If  $f$  is a measurable function finite almost everywhere on a set  $E \subset \mathbb{R}_n$ , then given any  $\varepsilon > 0$ , there is a function  $\phi$  continuous throughout  $\mathbb{R}_n$  such that*

$$\mu(E[f(x) \neq \phi(x)]) < \varepsilon.$$

In addition, if  $|f(x)| \leq M$  on the set  $E$ , one can require that the function  $\phi$  satisfy the inequality  $|\phi(x)| \leq M$  throughout  $\mathbb{R}_N$ .

Using Propositions A.1(3) and A.3 from Section A.2.9 and the corollaries of Theorem A.96, we can show that Luzin's theorem admits the following converse: if the conclusion of the theorem holds for a function  $f$  (which is not originally assumed to be measurable) defined on a set  $E \subset \mathbb{R}_n$  and finite almost everywhere on  $E$ , this function is measurable. The exact statement of the converse can be read, for instance, as follows: if a function  $f$  is defined on a measurable set  $E \subset \mathbb{R}_n$  and is finite almost everywhere on  $E$  and if for any  $\varepsilon > 0$  there exists a measurable subset  $E' \subset E$  on which  $f$  is continuous such that  $\mu(E \setminus E') < \varepsilon$ , then the function  $f$  is measurable on  $E$ . Thus, the property of measurable functions established in Theorem A.108 completely characterizes this class of functions. In other words, the structure of measurable functions in  $\mathbb{R}_n$  is closely connected with that of continuous functions.

The theorem below suggested by R. M. Fréchet can be obtained as a consequence of Luzin's theorem.

**Theorem A.109.** *If  $f$  is measurable and finite almost everywhere on a set  $E \subset \mathbb{R}_n$ , then there exists a sequence of functions  $\phi_k$  continuous throughout  $\mathbb{R}_n$  such that  $\phi_k(x) \rightarrow f(x)$  almost everywhere on  $E$ .*

From Theorem A.109 we can deduce the following theorem proved by the Italian mathematician G. Vitali (1875–1932).

**Theorem A.110.** *If  $f$  is measurable and finite almost everywhere on a set  $E \subset \mathbb{R}_n$ , it is equivalent to a function of at most the second Baire class defined throughout  $\mathbb{R}_n$  such that  $f(x) = g(x)$  for almost all  $X \in E$ .*

### A.2.11 The Lebesgue Integral of a Bounded Function

In this section we shall consider the Lebesgue integral in an arbitrary abstract measure space  $X$  with measure  $\mu$  for the special case of bounded functions, and the integrals in question will be taken only over sets with finite measure. As before, we shall denote by  $\mathfrak{G}$  the  $\sigma$ -algebra of subsets of  $X$  on which the measure  $\mu$  is defined.

Let a bounded function  $f$  be defined on a measurable set  $E \subset X$  with  $\mu(E) < +\infty$ . Breaking the set  $E$  in an arbitrary way into a finite number of disjoint measurable sets  $e_i (E = \bigcup_{i=1}^p e_i)$ , we define

$$M_i = \sup_{x \in e_i} f(x) \quad \text{and} \quad m_i = \inf_{x \in e_i} f(x), \quad i = 1, 2, \dots$$

The partition of  $E$  into the sets  $e_i$  of that type will itself be denoted by  $\tau$ . Now we form two sums,

$$S(\tau, f) = \sum_{i=1}^p M_i \mu(e_i), \quad s(\tau, f) = \sum_{i=1}^p m_i \mu(e_i),$$

which will be called respectively the *upper* and the *lower Lebesgue–Darboux sums*. For these sums we shall also use the shortened notation  $S(\tau)$  and  $s(\tau)$  when the function from which they are formed has already been defined.

Consider a partition  $\tau$  of the set  $E$  into sets  $e_i$  and another partition  $\tau'$  of  $E$  into sets  $e'_j$ . We shall say that the *partition  $\tau'$  is a refinement of the partition  $\tau$*  if each  $e'_j$  is contained in some  $e_i$ .

In the sequel we shall need the following two properties of Lebesgue–Darboux sums:

- (1) If the partition  $\tau'$  is a refinement of the partition  $\tau$ , then  $S(\tau') \leq S(\tau)$  and  $s(\tau') \geq s(\tau)$ .
- (2) No lower Lebesgue–Darboux sum  $s(\tau')$  exceeds any upper Lebesgue–Darboux sum  $S(\tau'')$  even if they are formed for two different partitions  $\tau'$  and  $\tau''$  of the set  $E$ .

It follows from property (2) that the set of all lower Lebesgue–Darboux sums  $s(\tau)$  corresponding to all the possible partitions  $\tau$  of the set  $E$  is bounded above by any upper sum. Consequently,

$$K = \sup_{\tau} s(\tau; f) \leq S(\tau; f).$$

Thus,  $K$  serves as a lower boundary for the set of upper sums, and therefore

$$L = \inf_{\tau} S(\tau; f) \geq K = \sup_{\tau} s(\tau; f).$$

**Definition A.61.** A function  $f$  is said to be *Lebesgue integrable* with respect to the given measure  $\mu$  (or briefly,  $\mu$ -integrable) on the set  $E$  if  $K = L$ ; in this case the common value of the numbers  $K$  and  $L$  is called the *Lebesgue integral* of the function  $f$  over the set  $E$  and is denoted by  $\int_E f d\mu$  or (sometimes)  $\int_E f(x) d\mu$ .

If  $E$  is a segment  $[a, b]$  belonging to  $\mathbb{R}_1$  and  $\mu$  is the Lebesgue measure, the classical notation  $\int_a^b f(x) dx$  or  $\int_a^b f dx$  is also used.

**Theorem A.111.** *Every bounded function  $f$  measurable on a set  $E$  is integrable on  $E$ .*

An important special case of the notion of the integral defined above appears when  $f$  is a function in a Euclidean space with the Lebesgue measure. From Theorem A.95 and from Theorem A.111 above we deduce the following corollary for functions in a Euclidean space.

**Corollary A.9.** *If a function  $f$  is continuous on a bounded closed set  $E \subset \mathbb{R}_n$ , then  $f$  is Lebesgue integrable on  $E$ .*

Now we establish a number of simple properties of the Lebesgue integral. As a rule, in what follows, integrand functions are assumed to be bounded and measurable, and the integrals are taken over sets of finite measure. It should be noted that if  $\mu E = 0$ , then any bounded function  $f$  defined on  $E$  is integrable on  $E$ . This property is a trivial consequence of the definition of the integral. We shall also agree that the integral of any function taken over the empty set makes sense and is understood as equal to zero.

**Theorem A.112 (Estimation of the integral).** *If  $C \leq f(x) \leq D$  and  $f$  is measurable on a set  $E$ , then*

$$C\mu(E) \leq \int_E f d\mu \leq D\mu(E). \quad (\text{A.26})$$

**Corollary A.10.** (1) *If  $f(x) \geq 0$  on  $E$  then  $\int_E f d\mu \geq 0$ .*

(2) *If  $f(x)$  is equal to a constant  $p$  on  $E$ , then  $\int_E f d\mu = p \times \mu(E)$ .*

**Theorem A.113 ( $\sigma$ -additivity of the integral).**

*Let  $E$  be a set representable as a finite or countable union of pairwise disjoint measurable sets  $E_j$  ( $E = \bigcup_j E_j$ ). Then for every function  $f$  bounded and measurable on the set  $E$  we have*

$$\int_E f d\mu = \sum_j \int_{E_j} f d\mu. \quad (\text{A.27})$$

This theorem implies that the integral  $\int_e f d\mu$  of a given (fixed) function  $f$  is a  $\sigma$ -additive set function defined on the  $\sigma$ -algebra  $\mathfrak{G}_E$  of all measurable subsets  $e \in \mathfrak{G}_E$ , that is, in this case the integral is a (new) measure defined on  $\mathfrak{G}_E$ .

**Corollary A.11.** (1) *If  $E = \bigcup_{j=1}^p E_j$ , where  $E_j$  are measurable and pairwise disjoint and  $f(x) = c_j$  on  $E_j$ ,  $j = 1, 2, \dots, p$ , where the  $c_j$  are constants, then*

$$\int_E f d\mu = \sum_{i=1}^p c_i \mu(E_i).$$

(2) *If  $E = \bigcup_{m=1}^{\infty} E'_m$ , where the sets  $E'_m$  are measurable and form an increasing sequence, then*

$$\int_E f d\mu = \lim \int_{E'_n} f d\mu.$$

(3) *If  $f$  is bounded, measurable, and nonnegative on  $E$  ( $f(x) \geq 0$ ) and  $E'$  is a measurable subset of  $E$ , then*

$$\int_{E'} f d\mu \leq \int_E f d\mu.$$

**Theorem A.114.** For any two functions  $f$  and  $g$  bounded and measurable on  $E$  and any constant  $c$  we have

$$\int_E (f \pm g) d\mu = \int_E f d\mu \pm \int_E g d\mu, \quad (\text{A.28})$$

$$\int_E cf d\mu = c \int_E f d\mu \quad (\text{A.29})$$

**Theorem A.115.** For any bounded measurable function  $f$  we have the inequality

$$\left| \int_E f d\mu \right| \leq \int_E |f| d\mu.$$

**Theorem A.116.** If  $f \sim g$  on a set  $E$  (both functions are supposed to be bounded and measurable) then

$$\int_E f d\mu = \int_E g d\mu. \quad (\text{A.30})$$

*Remark A.9.* More-general propositions can be proved: if  $f$  and  $g$  are bounded on a set  $E$ ,  $f \sim g$  on  $E$ , and  $f$  is integrable on that set, then the function  $g$  is also integrable on  $E$  and the equality (A.30) holds.

*Remark A.10.* The theorems above mean that any variation of the values of the integrand function on a set of measure zero does not affect the value of the integral.

**Theorem A.117.** If  $f(x) \geq 0$  almost everywhere on  $E$  ( $f$  is assumed to be bounded and measurable) and  $\int_E f d\mu = 0$  then  $f \sim 0$ .

**Theorem A.118 (Termwise integration of inequalities).** If  $f(x) \leq g(x)$  almost everywhere on a set  $E$  (both functions are bounded and measurable) then

$$\int_E f d\mu \leq \int_E g d\mu.$$

Let us consider a set  $E \subset X$  with  $\mu(E) < +\infty$  and a sequence of bounded measurable functions  $f_k, k = 1, 2, \dots$ , defined on  $E$ . Suppose that  $f_k \implies f$ , where the function  $f$  is also bounded (and, of course, measurable). If for such a sequence the limit relation

$$\int_E f_k d\mu \rightarrow \int_E f d\mu \quad (\text{A.31})$$

holds, we say that for the given sequence of functions on the set  $E$  it is allowable to pass to the limit under the integral sign.

*Remark A.11.* Property (A.31) may not hold even when  $f_k(x) \rightarrow f(x)$  at all points  $x \in E$ .

**Theorem A.119 (H. Lebesgue).** *Let  $f_k$ ,  $k = 1, 2, \dots$ , be bounded and measurable functions on a set  $E \subset X$  (with  $\mu(E) < +\infty$ ) and let there exist a constant  $M > 0$  such that  $|f_k(x)| \leq M$  for all  $k$  and for almost all  $x$  on  $E$ . If  $f_k \implies f$  on  $E$  and the function  $f$  is bounded on the set  $E$ , then relation (A.31) holds.*

### A.2.12 The Space $\mathbb{S}$ of Measurable Functions

Let  $E \subset X$  and  $\mu(E) < +\infty$ . By  $\mathbb{S}$  we shall denote the set of all measurable functions  $f$  defined on  $E$  that are finite almost everywhere. We shall agree that equivalent functions are identified with each other, that is, they are regarded as a single element of the set  $\mathbb{S}$ . Thus, every element  $f$  of  $\mathbb{S}$  is represented not by one particular function but by the corresponding class of all mutually equivalent functions. In particular, every element  $f \in \mathbb{S}$  can be represented by a function assuming finite values everywhere on  $E$ , and thus we can think of  $\mathbb{S}$  as consisting of only such functions. Given two arbitrary elements  $f, g \in \mathbb{S}$ , we define the distance between them by the formula

$$\rho(f, g) = \int_E \frac{|f - g|}{1 + |f - g|} d\mu. \quad (\text{A.32})$$

The integrand being bounded and measurable, the integral always exists. If  $f$  or  $g$  is replaced by an equivalent function belonging to  $\mathbb{S}$ , the distance  $\rho(f, g)$  does not change. Moreover, the function  $\rho(f, g)$  satisfies all the axioms of distance in a metric space. Thus, the set  $\mathbb{S}$ , equipped with the metric (A.32), is a metric space.

**Theorem A.120.** *Convergence in distance in  $\mathbb{S}$  coincides with convergence in measure.*

**Theorem A.121.**  *$\mathbb{S}$  is a complete metric space.*

**Theorem A.122.** *For any countable set of nonnegative functions  $f_k \in \mathbb{S}$ ,  $k = 1, 2, \dots$ , there exist numbers  $\alpha_k > 0$  and a function  $h \in \mathbb{S}$  such that  $\alpha_k f_k(x) \leq h(x)$  almost everywhere on  $E$  for any  $k$ .*

Now we can strengthen Theorem A.106 on convergence regulators.

**Theorem A.123 (Existence of common convergence regulators).** *If there are functions*

$$f_k^{(p)} \in \mathbb{S} \quad (k, p = 1, 2, \dots)$$

*such that  $f_k^{(p)}(x) \xrightarrow[k \rightarrow \infty]{} 0$  for each  $p$  almost everywhere on  $E$ , then there exists a common convergence regulator for all the sequences  $\{f_k^{(p)}\}$ .*



### A.2.13 Summable Functions

In this section we shall generalize the concept of the Lebesgue integral to the case that the integrand can be an unbounded function and the domain of integration can have an infinite measure ( $\sigma$ -finite).

**Definition A.62.** Let a function  $f(x) \geq 0$  be measurable and finite almost everywhere on a set  $E \subset X$ . Taking all the possible measurable subsets  $e \subset E$  having a finite measure on which the function  $f$  is bounded, we set

$$\int_E f d\mu = \sup_e \int_e f d\mu. \quad (\text{A.33})$$

We thus have  $\int_E f d\mu \geq 0$ . Then if  $\int_E f d\mu < +\infty$ , the function  $f$  is said to be *summable with respect to the measure  $\mu$*  or simply  *$\mu$ -summable on the set  $E$* .

If the function  $f(x) \geq 0$  is measurable on  $E \subset X$  but  $\mu\{x \in E | f(x) = +\infty\} > 0$ , we shall say by definition that  $\int_E f d\mu = +\infty$ .

If  $f(x) \leq 0$  and  $f$  is measurable on  $E \subset X$ , then we set

$$\int_E f d\mu = - \int_E |f| d\mu.$$

We say that the function  $f$  is summable if the integral  $\int_E |f| d\mu$  assumes a finite value, that is, if the function  $|f|$  is summable.

Let  $f$  be measurable on a set  $E$ . Putting

$$E_1 = \{x \in E : f(x) \geq 0\}, \quad E_2 = \{x \in E : f(x) < 0\},$$

we thus break  $E$  into two measurable sets. Now the integral of  $f$  over the set  $E$  is disjoint by the formula

$$\int_E f d\mu = \int_{E_1} f d\mu + \int_{E_2} f d\mu = \int_{E_1} |f| d\mu - \int_{E_2} |f| d\mu. \quad (\text{A.34})$$

It should be taken into account that formula (A.34) is meaningful only if at least one of the integrals  $\int_{E_1} f d\mu$  and  $\int_{E_2} f d\mu$  is finite. If both integrals are infinite, that is,  $\int_{E_1} f d\mu = +\infty$  and  $\int_{E_2} f d\mu = -\infty$ , the integral  $\int_E f d\mu$  makes no sense.

The function  $f$  is called summable on a set  $E$  if  $f$  is summable on each of the sets  $E_1$  and  $E_2$ , that is, if  $\int_E f d\mu$  has a finite value.

Apparently, if a function is summable on a set  $E$ , it is also summable on every measurable subset. If  $\mu(E) = 0$ , then every measurable function  $f$  defined on  $E$  is summable, and  $\int_E f d\mu = 0$ . If the function  $f$  is a *bounded measurable function* on the set  $E$  with  $\mu(E) < +\infty$ , then it is summable on this set.

We now list some properties of the integral. All the functions below are supposed to be finite almost everywhere, and we do not mention this condition in the sequel.

**Theorem A.126 (Absolute summability).** *In order that a function  $f$ , measurable on a set  $E \subset X$ , be summable on that set, it is necessary and sufficient that the function  $|f|$  be summable on  $E$ .*

**Theorem A.127.** *If the integral  $\int_E f d\mu$  exists, then*

$$\left| \int_E f d\mu \right| \leq \int_E |f| d\mu. \quad (\text{A.35})$$

**Theorem A.128 (Absolute continuity).** *Let  $f$  be summable on a set  $E$ . Then given any  $\varepsilon > 0$ , there is  $\delta > 0$  such that*

$$\left| \int_E f d\mu \right| < \varepsilon \quad \text{if } E' \subset E \text{ and } \mu(E') < \delta.$$

**Theorem A.129 ( $\sigma$ -additivity).** *Let the set  $E$  be representable as a finite or countable union of disjoint measurable sets  $E_j$  ( $E = \bigcup_j E_j$ ). Then:*

(1) *If the integral  $\int_E f d\mu$  exists, then each of the integrals  $\int_{E_j} f d\mu$  also exists and*

$$\int_E f d\mu = \sum_j \int_{E_j} f d\mu. \quad (\text{A.36})$$

(2) *If  $f$  is summable on each  $E_j$ , then  $f$  is summable on  $E$  if and only if the condition*

$$\sum_j \int_{E_j} |f| d\mu < +\infty \quad (\text{A.37})$$

*holds.*

**Corollary A.12.** *If*

$$E = \bigcup_{j=1}^{\infty} E_j,$$

*where the sets  $E_j$  are measurable and disjoint,  $\mu(E_j) < +\infty$  for all  $j$ ,  $f(x) = c_j$  on  $E_j$ ,  $j = 1, 2, \dots$ , and the  $c_j$  are constants and  $\sum_{j=1}^{\infty} |c_j| \mu(E_j) < +\infty$ , then  $f$  is summable on  $E$  and*

$$\int_E f d\mu = \sum_{j=1}^{\infty} c_j \mu(E_j).$$

*This is a generalization of Corollary A.11(1) to Theorem A.113.*

The  $\sigma$ -additivity of the integral proved above implies that if measurable sets  $E_p$ ,  $p = 1, 2, \dots$ , form an increasing sequence,  $E = \bigcup_{p=1}^{\infty} E_p$ , and  $\int_E f d\mu$  exists then

$$\int_E f d\mu = \lim \int_{E_p} f d\mu. \quad (\text{A.38})$$

**Theorem A.130.** *If  $f$  and  $g$  are measurable on a set  $E$ ,  $f \sim g$ , and at least one of the integrals  $\int_E f d\mu$  and  $\int_E g d\mu$  exists, the other integral also exists and*

$$\int_E f d\mu = \int_E g d\mu.$$

*In particular, if  $f$  is summable on  $E$ , then  $g$  is also summable.*

**Theorem A.131.** *If  $f(x) \geq 0$  almost everywhere on  $E$  and  $\int_E f d\mu = 0$ , then  $f \sim 0$ .*

**Theorem A.132.** *If  $f$  and  $g$  are measurable on a set  $E$ ,  $g$  is summable on  $E$ , and  $|f(x)| \leq |g(x)|$  almost everywhere on  $E$ , the function  $f$  is also summable and*

$$\left| \int_E f d\mu \right| \leq \int_E g d\mu.$$

**Theorem A.133.** *For any two functions  $f$  and  $g$  summable on a set  $E$  and for any constant  $c$ , the functions  $f \pm g$  and  $cf$  are also summable and*

$$\int_E (f \pm g) d\mu = \int_E f d\mu \pm \int_E g d\mu, \quad (\text{A.39})$$

$$\int_E cf d\mu = c \int_E f d\mu. \quad (\text{A.40})$$

**Theorem A.134.** *If  $f$  and  $g$  are summable on a set  $E$  and  $f(x) \leq g(x)$  almost everywhere on  $E$  then*

$$\int_E f d\mu \leq \int_E g d\mu. \quad (\text{A.41})$$

**Theorem A.135 (H. Lebesgue).** *Let  $\{f_k\}$ ,  $k = 1, 2, \dots$ , be a sequence of functions summable on a set  $E \subset X$  that converges in measure to a function  $f$ , i.e.,  $f_k \Leftarrow f$ . If there exists a nonnegative summable function  $\phi$  such that  $|f_k(x)| \leq \phi(x)$  for each  $k$  almost everywhere on  $E$ , then the function  $f$  is also summable and*

$$\lim \int_E f_k d\mu = \int_E f d\mu. \quad (\text{A.42})$$

*Remark A.12.* 1. Theorem A.119 is a special case of Theorem A.135.

2. Lebesgue's theorem holds when in its statement convergence in measure is replaced by convergence almost everywhere (to a measurable function).

There is a stronger theorem for a monotone sequence of functions. We shall state it here for a monotone sequence of nonnegative functions.

**Theorem A.136 (B. Levi).** *If there is a nondecreasing sequence of functions  $f_k(x) \geq 0$ ,  $k = 1, 2, \dots$ ,  $f_k(x) \leq f_{k+1}(x)$ , for all  $k$  measurable on the set  $E$  and  $f_k(x) \rightarrow f(x)$  for all  $x \in E$ , where  $f$  is finite almost everywhere on  $E$ , then equality (A.42) holds:*

$$\lim \int_E f_k d\mu = \int_E f d\mu.$$

Let us introduce another useful notion. If  $f$  is a nonnegative function measurable on a set  $E$ , we define, for every natural number  $m$  and every  $x \in E$ , the *truncated function*

$$f_m(x) = \begin{cases} f(x), & \text{if } f(x) \leq m, \\ m, & \text{if } f(x) > m. \end{cases}$$

It is clear that all  $f_m$ ,  $m = 1, 2, \dots$ , are bounded and measurable on  $E$ .

We give here some consequences of Levi's theorem:

- (1) For any nonnegative measurable function  $f(x)$  and the corresponding sequence of truncated functions  $f_m(x)$ ,  $m = 1, 2, \dots$ , it is allowable to pass to the limit under the integral sign.
- (2) If in addition to the condition of Levi's theorem, we suppose that the integrals  $\int_E f_k d\mu$ ,  $k = 1, 2, \dots$ , are uniformly bounded, this will imply that the limit function  $f$  is finite almost everywhere.
- (3) If  $\{u_k(x)\}$  is a sequence of nonnegative functions ( $u_k(x) \geq 0$ ,  $k = 1, 2, \dots$ ), summable on a set  $E$ , and  $\sum_{k=1}^{\infty} \int_E u_k d\mu > \infty$ , then the function  $s(x) = \sum_{k=1}^{\infty} u_k(x)$  is finite almost everywhere on  $E$  and

$$\int_E s d\mu = \sum_{k=1}^{\infty} \int_E u_k d\mu.$$

Our next theorem establishes a relation slightly weaker than equality (A.42).

**Theorem A.137 (P. Fatou).** *Let a sequence of functions*

$$f_k(x) \geq 0, \quad k = 1, 2, \dots,$$

*measurable and finite almost everywhere, be defined on a set  $E$ . If this sequence converges in measure on  $E$ , then*

$$\int_E f d\mu \leq \sup_k \int_E f_k d\mu. \tag{A.43}$$

*Remark A.13.* Fatou's theorem also applies to the case in which

$$f_k(x) \rightarrow f(x)$$

almost everywhere on  $E$  and is measurable. In addition, in the case of convergence almost everywhere, the requirement that the functions  $f_k$  and  $f$  be finite almost everywhere can be omitted in the formulation of the theorem.

### A.2.14 The Space $L$ of Summable Functions

Denote the collection of all functions defined and summable on a fixed set  $E \subset X$  by  $L$ . To indicate, if necessary, the domain of definition of the functions we shall also write  $L_E$ . As in the definition of the space  $\mathbb{S}$  we shall identify in  $L$  functions equivalent to each other; this will allow us to regard every element of the set  $L$  as represented by a function assuming finite values everywhere.

Theorem A.133 shows that  $L$  is a linear space. To introduce a norm in the space  $L$ , we put

$$\|f\| = \int_E |f| d\mu$$

for any function  $f$ .

The relation  $f_k \rightarrow f$  in  $L$  is understood as convergence in norm and means that

$$\int_E |f_k - f| d\mu \rightarrow 0.$$

Such a sequence  $f_k$  is said to converge to  $f$  in the mean of order one.

**Theorem A.138.** *If  $f_k \rightarrow f$  in norm in the space  $L$ , then  $f_k \Leftarrow f$  (that is, mean convergence implies convergence in measure to the same limit function).*

**Theorem A.139.**  *$L$  is a Banach space.*

In the case that the space  $L$  consists of summable functions defined on the set  $E \subset \mathbb{R}_n$ ,  $\mu$  being the Lebesgue measure, the space  $L$  is separable.

### A.2.15 Variations of Additive Set Functions

Let  $X$  be an abstract set and  $\mathfrak{G}$  an algebra of some of its subsets. We shall consider an additive set function  $\varphi$  defined for all  $A \in \mathfrak{G}$ . We shall agree that the infinite values of  $\varphi$  can be of only one sign. We shall also assume that  $\varphi(\emptyset) = 0$ .

**Definition A.63.** For a given function  $\varphi$ , the set functions expressed in terms of  $\varphi$  by means of the formulas

$$\begin{aligned}\varphi_+(A) &= \sup_{B \subset A, B \in \mathfrak{G}} \varphi(B), \\ \varphi_-(A) &= \sup_{B \subset A, B \in \mathfrak{G}} \{-\varphi(B)\}, \\ |\varphi|(A) &= \varphi_+(A) + \varphi_-(A) \quad (A \in \mathfrak{G}),\end{aligned}$$

are called the *positive variation*, the *negative variation*, and the *total variation* of  $\varphi$ .

It should be noted that if the function  $\varphi$  is bounded, that is, if there exists a constant  $K$  such that  $|\varphi(A)| \leq K$  for all  $A \in \mathfrak{G}$ , then all the three variations of  $\varphi$  have finite values. If  $\varphi$  is bounded only from above (below), then the variation  $\varphi_+$  ( $\varphi_-$ ) is finite. The converses of these assertions are also true.

*Remark A.14.* Any nonnegative additive set function defined on algebra that assumes only finite values is necessarily bounded. This follows from its monotonicity.

**Theorem A.140.** *All three variations  $\varphi_+$ ,  $\varphi_-$ , and  $|\varphi|$  are nonnegative additive set functions on the algebra  $\mathfrak{G}$ . If  $\varphi(A)$  is finite for a set  $A \in \mathfrak{G}$ , then*

$$|\varphi|(A) = \sup_{\substack{B_1, B_2 \subset A \\ B_1, B_2 \in \mathfrak{G}}} \{\varphi(B_1) - \varphi(B_2)\}. \quad (\text{A.44})$$

**Theorem A.141.** *Every additive set function  $\varphi$  defined on the algebra  $\mathfrak{G}$  bounded from above or from below is representable as the difference of two nonnegative additive set functions, namely*

$$\varphi = \varphi_+ - \varphi_-. \quad (\text{A.45})$$

Relation (A.45) is referred to as the *Jordan decomposition* of  $\varphi$ .

The total variation of the function  $\varphi$  can also be determined with the aid of the formula

$$|\varphi|(A) = \sup \sum_{i=1}^n |\varphi(A_i)|, \quad (\text{A.46})$$

where the supremum is taken over all finite partitions of the set  $A$  into disjoint sets  $A_i \in \mathfrak{G}$ .

Formula (A.46) suggests another approach to the definition of the positive, negative, and total variations of a bounded additive set function. In this case we can first define the total variation by means of formula (A.46), and then the positive and negative variations can be introduced according to the formulas

$$\varphi_+ = \frac{1}{2}(|\varphi| + \varphi), \quad \varphi_- = \frac{1}{2}(|\varphi| - \varphi).$$

**Theorem A.142.** *If a set function  $\varphi$  is  $\sigma$ -additive on the algebra  $\mathfrak{G}$ , then all three variations  $\varphi_+$ ,  $\varphi_-$ , and  $|\varphi|$  are also  $\sigma$ -additive.*

In the next two theorems, we assume that the domain of definition  $\mathfrak{G}$  of  $\varphi$  is a  $\sigma$ -algebra.

**Theorem A.143.** *If the function  $\varphi$  takes finite values and is  $\sigma$ -additive on the algebra  $\mathfrak{G}$ , it is bounded.*

**Corollary A.13.** *If a  $\sigma$ -additive set function  $\varphi$  defined on a  $\sigma$ -algebra  $\mathfrak{G}$  is  $\sigma$ -finite, then its total variation  $|\varphi|$  and together with it the variations  $\varphi_+$  and  $\varphi_-$  are also  $\sigma$ -finite.*

**Theorem A.144.** *If a  $\sigma$ -additive function  $\varphi$  defined on a  $\sigma$ -algebra  $\mathfrak{G}$  does not assume the value  $+\infty$  ( $-\infty$ ), it is bounded from above (below).*

### A.2.16 The Radon Integral

The basic idea of the generalization of the notion of integral suggested by J. Radon is that integration with respect to a measure is replaced by integration with respect to an arbitrary  $\sigma$ -additive set function (which can assume negative values as well).

Denote by  $X$  an abstract set and consider a  $\sigma$ -algebra  $\mathfrak{G}$  of its subsets regarded, by definition, as measurable and a  $\sigma$ -additive and  $\sigma$ -finite set function  $\varphi$  defined on  $\mathfrak{G}$ . From the previous section it follows (Theorem A.142 and Corollary A.13 to Theorem A.143) that  $\varphi_+$  and  $\varphi_-$  are  $\sigma$ -finite measures in  $X$ . Furthermore, since  $\varphi$  can assume infinite values of only one sign, Theorem A.144 tells us that it is bounded from at least one side, and therefore equality (A.45) applies to it.

**Definition A.64.** Let  $f$  be a measurable function defined on a measurable set  $E \subset X$ . The Radon integral of the function  $f$  with respect to the function  $\varphi$  over the set  $E$  is defined by means of the formula

$$\int_E f d\varphi = \int_E f d\varphi_+ - \int_E f d\varphi_- \quad (\text{A.47})$$

under the assumption that the integrals on the right-hand side of the formula as well as their difference make sense (that is, either at least one of integrals on the right-hand side is finite or one of them is equal to  $+\infty$  while the other is equal to  $-\infty$ ). Otherwise, the integral  $\int_E f d\varphi$  does not exist. If the integral  $\int_E f d\varphi$  has a finite value, the function  $f$  is said to be Radon summable on  $E$  with respect to the function  $\varphi$ .

In contrast to the integral with respect to a measure, the Radon integral  $\int_E f d\varphi$  of a nonnegative measurable function  $f$  may not exist. In particular, the fact that  $\int_E f d\varphi$  exists no longer implies that  $\int_E |f| d\varphi$  also exists. At the same time,  $f$  is summable with respect to  $\varphi$  if and only if the function  $f$  and together with it the modulus  $|f|$  are summable with respect to the total variation  $|\varphi|$ .

The Radon integral is sometimes also referred to as the *Lebesgue–Stieltjes integral*.

From Definition A.64 of the Radon integral, it readily follows that most of the properties of the Lebesgue integral from Section A.2.13 extend to the Radon integral word for word or with minor changes. In particular, the integral remains  $\sigma$ -additive and Theorem A.133 holds. Formula (A.47) directly implies the following estimate for the Radon integral:

$$\left| \int_E f d\varphi \right| \leq \int_E |f| d|\varphi|. \quad (\text{A.48})$$

It follows that the Radon integral of a summable function possesses the property of absolute continuity (see Theorem A.133 with respect to the measure  $|\varphi|$ ). The theorems involving the notion of equivalent functions and relations that hold “almost everywhere” is understood relative to the measure  $|\varphi|$ : “almost everywhere on  $E$ ” should mean “for all  $x \in E$  with a possible exception of the points of a subset  $E'$  for which  $|\varphi|(E') = 0$ .”

### A.2.17 Integration of Finitely Additive Set Functions

Here we shall limit ourselves to the case that both the integrand function and the function with respect to which the integration is carried out are bounded. Thus, let  $X$  be an abstract set and  $\mathfrak{G}$  an algebra of some of its subsets. We shall consider a bounded additive set function  $\varphi$  defined on  $\mathfrak{G}$ . Let us begin with the case that  $\varphi$  is nonnegative.

Taking a real point function  $f$  defined and bounded on a set  $E \subset \mathfrak{G}$  and an arbitrary partition  $\tau$  of the set  $E$  into a finite number of disjoint sets  $e_i \in \mathfrak{G}$ ,  $i = 1, 2, \dots, p$ , we form the Lebesgue–Darboux sums

$$\begin{aligned} s(\tau; f) &= \sum_{i=1}^p m_i \varphi(e_i), \\ S(\tau; f) &= \sum_{i=1}^p M_i \varphi(e_i), \\ m_i &= \inf_{x \in e_i} f(x), \\ M_i &= \sup_{x \in e_i} f(x) \quad (i = 1, 2, \dots, p). \end{aligned}$$

The properties of the Lebesgue–Darboux sums established in Section A.2.11 continue to hold in the case that the function  $\varphi$  is finitely additive. We shall define the integral  $\int_E f d\varphi$  by analogy with the Lebesgue integral (Section A.2.11), that is, as the common value of  $\sup_{\tau} s(\tau; f)$  and  $\inf_{\tau} S(\tau, f)$  in case the supremum and the infimum coincide.

In the general case that  $\varphi$  can assume values of any sign, the integral  $\int_E f d\varphi$  is defined by means of formula (A.47) under the assumption that both integrals on the right-hand side exist.

As in Section A.2.11, it can be proved that *if a bounded function  $f$  is  $\mathfrak{G}$ -measurable (as usual, the  $\mathfrak{G}$ -measurability of  $f$  means that all the Lebesgue sets of the function  $f$  are contained in the algebra  $\mathfrak{G}$ ), then it is integrable.* But in this case, the class of integrable functions can be essentially wider than that of measurable functions.

The integral with respect to a finitely additive set function has much in common with the Radon integral. It can easily be shown that if  $f$  is  $\varphi$ -integrable, then  $f$  and  $|f|$  are  $|\varphi|$ -integrable ( $|\varphi|$  is the total variation of  $\varphi$ ) and inequality (A.48) holds. Furthermore, this inequality implies that if

$$|f(x)| \leq K,$$

then

$$\left| \int_E f d\varphi \right| \leq K |\varphi|(E).$$

Instead of the  $\sigma$ -additivity, in this case we can prove only the finite additivity of the integral. For bounded  $\varphi$ -integrable functions  $f$  and  $g$  we also have Theorem A.114.

Let us consider the special case in which  $X = \mathbb{R}_n$ , the algebra  $\mathfrak{G}$  contains all closed sets,  $E$  is a bounded closed set belonging to  $\mathbb{R}_n$ , and  $f$  is continuous on  $E$ . In this case, the integral  $\int_E f d\varphi$  exists for any bounded additive set function  $\varphi$  on  $\mathfrak{G}$ .

The same result can be obtained in the case that  $\mathfrak{G}$  is the algebra of subsets of  $\mathbb{R}_n$  generated by the system of all cells (see Section A.2.7).

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